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COLLEGE ALGEBRA

BY

H. L. RIETZ
UNIVERSITY OF IOWA

AND

A. R. CRATHORNE
UNIVERSITY OF ILLINOIS

FOURTH EDITION

NEW YORK
HENRY HOLT AND COMPANY

6–45

A WARTIME BOOK
THIS COMPLETE EDITION IS PRODUCED IN FULL COMPLIANCE WITH THE GOVERNMENT'S REGULATIONS FOR CONSERVING PAPER AND OTHER ESSENTIAL MATERIALS.

PRINTED IN THE
UNITED STATES OF AMERICA

PREFACE

This book is designed primarily for use as a text-book in the freshman year of colleges and technical schools. Special attention is directed to the following features:

(1) The method of reviewing the algebra of the secondary schools.

(2) The selection and omission of material.

(3) The explicit statement of assumptions upon which the proofs are based.

(4) The application of algebraic methods to physical problems.

For the majority of college freshmen, a considerable period of time elapses between the completion of the high school algebra and the beginning of college mathematics. The review of the secondary school algebra is written for these students. This part of the book is, however, more than a hasty review. While the student is reviewing a first course, he is at the same time making a distinct advance by seeing the subject-matter from new viewpoints, which his added maturity enables him to appreciate. For example, the functional notation, graphs, and determinants are introduced and used to advantage in the review. The extensions of the number concept receive fuller treatment than is usual in a college algebra. The various classes of numbers from positive integers to complex numbers are treated in the order in which they are demanded by the equation.

The application of algebra in the more advanced courses in mathematics has been an important factor in determining the subject-matter. Not only are some of the topics usually treated in the traditional course in algebra entirely omitted, but in each chapter the material is restricted to the development of those central points which experience has shown to be essential. While a complete discussion of limits and infinite series does not properly belong in a course in algebra, it has been thought best to include an introduction to these subjects which covers in considerable detail only the theory necessary to a discussion of the comparison and ratio tests. From the experience of the authors, a great deal is gained by thus taking a very elementary first cours in limits and series.

While it is out of place in a book of this character to attemp a critical study of fundamentals, great care has been taken t point out just what is proved and what is assumed in so far as first-year student can be expected to appreciate the necessity o assumptions.

Without trying to teach physics or engineering, many problems are introduced in which the principles of algebra are applied t physical problems, but no technical knowledge is assumed on the part of the student. Rules for the mechanical guidance of students in solving problems have been used sparingly.

The authors take great pleasure in acknowledging their indebtedness to their colleagues in the mathematical and engineering departments of the University of Illinois. We are indebted to Professors Haskins and Young for suggestions during the preparation of the manuscript as well as for a critical reading of the manuscript; to Professors Townsend, Goodenough, Miller Wilczynski, Dr. Lytle, and to Professor Kuhn of Ohio State University for suggestions upon the manuscript; to Professor Watson for some of the practical problems; and again to Professor Goodenough for assistance in seeing the book through the press.

H. L. RIETZ
A. R. CRATHORNE

PREFACE TO THE FOURTH EDITION

This Fourth Edition is the most painstaking revision that t book has undergone in its thirty years of classroom use. Wh the general character and style of the book remain essentially u altered, many features have been changed in the light of the teac ing experience of the authors and in response to the suggestions many other teachers and to the changing times.

The number of easy exercises has been greatly increased. Sh lists of oral exercises have been added as a teaching device in introduction of certain new ideas to the students.

The exercises and problems have been very completely chan except in the case of the rather unique problems that have bee leading characteristic of the book.

The treatment of the laws of exponents and radicals has been moved from Chapters I and II to a Chapter VI so as to allow the earlier introduction of chapters on graphic representation and simultaneous linear equations.

The new Chapter I is only part of old Chapter I and has been given over entirely to material to be used by those teachers who wish to stimulate thought at the very beginning of the course on the theoretical side of elementary algebra. Those who prefer to assume the facts of Chapter I without comment should begin with Chapter II, which is a review of the elementary operations of pre-college algebra.

The chapters on systems of linear equations, systems of equations involving quadratics, and the theory of equations have been much modified to make them more teachable.

The chapter on Compound Interest and Annuities is presented as a chapter in a college algebra, not as a chapter in a text-book on Mathematics of Finance, where extended tables are available for the use of the student. No tables are given in this chapter but in connection with many of the exercises and problems the necessary tabular values are given.

As in the Third Edition, the subject of probability is presented mainly from the viewpoint of statistical probability instead of putting nearly all the emphasis on deductive probability in relation to games of chance.

Following the practice in the Third Edition, we give the answers to exercises and problems with odd numbers only. An answer book is, however, available giving the answers to the even numbered exercises and problems.

The thanks of the authors are again due to many teachers who have contributed valuable suggestions, especially to Professor Roscoe Woods of the University of Iowa for a critical reading of the manuscript of the Fourth Edition, and to Dr. G. E. Moore of the University of Illinois and Professor S. E. Brasefield of Rutgers University for suggestions during the preparation of the revision.

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INFINITE SERIES

A LIST OF SIGNS AND SYMBOLS

$+$, read *plus*. $-$, read *minus*.
$\times$, or $\cdot$, read *times*.
$\div$, read *divided by*.
$=$, read *is equal to*.
$\equiv$, read *is identical with*.
$\neq$, read *is not equal to*.
$\rightarrow$, read *approaches*.
$<$, read *is less than*.
$>$, read *is greater than*.
$\leqq$, read *is less than or equal to*.
$\geqq$, read *is greater than or equal to*.
$a!$ or $\underline{|a}$, read *factorial a*.

( )	Parentheses.	Signs of aggregation. These signs are used to collect together symbols which are to be treated in operations as one symbol.
[ ]	Brackets.	
{ }	Braces.	
——	Vinculum.	
|	Bar.	

a_r, read *a subscript r*, or *a sub r*.
x', x'' . . . , read *x prime*, *x second* . . . respectively.
$\lim x$, read *limit of x*.
$x \rightarrow \infty$, read *x becomes infinite*, or *x increases beyond bound*.
$\log_a n$, read *logarithm of n to the base a*.
$|a|$, read *absolute value of a*.
a^n, read *a to the nth power*, or *a exponent n*.
$\sqrt{a}$, read *square root of a*.
$\sqrt[n]{a}$, read *nth root of a*.
$f(x)$, $\phi(x)$, etc., read "*f*" *function of x*, "ϕ" *function of x, etc.*
$P(n, r)$ read *number of permutations of n things taken r at a time.*
$C(n, r)$ read *number of combinations of n things taken r at a time.*
(x, y), read *point whose coördinates are x and y.*

COLLEGE ALGEBRA

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ s	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ϵ	Epsilon	Ν ν	Nu	Φ ϕ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

CHAPTER I *

INTRODUCTION

1. Numbers. In counting the objects of a group the child makes his first acquaintance with numbers. These are the numbers called **positive integers.**

He next employs the number we call a **rational fraction,** probably thinking of it first as part of a whole and later as the quotient of two integers.

Perhaps in the fall of a thermometer below zero, the student had his first experience in the use of **negative numbers,** even if he was not taught to use the word negative. He may also come early to the convenient use of the negative number to represent debit when the corresponding positive number means credit.

To express the length of the diagonal of a square of side one unit, or to find a number which multiplied by itself gives some integer, not a perfect square, say 2, he uses a number which is neither an integer nor a rational fraction, and employs a radical sign to represent it by writing $\sqrt{2}$ where $\sqrt{2} \times \sqrt{2} = 2$. Such numbers belong to a class of numbers known as **irrational numbers.** (See p. 63.)

2. Graphical representation of real numbers. The four classes of numbers mentioned in Art. 1 belong to the so-called "real numbers" used in arithmetic and algebra. They may be represented by the points of a straight line as follows: Let $X'X$ be this line. (Fig. 1.) Choose a point O on this line and call it the zero point or origin. Adopt some unit of measurement OA.

Beginning at O and proceeding in both directions, apply the unit of measure to mark OX and OX' at equal intervals, thus forming a scale of indefinite length of which a part is shown in Fig. 1. The positive and negative integers may then be conveniently represented by the points marking the intervals. Simi-

* Teachers who wish to emphasize the theoretical side of elementary algebra should begin the course with Chapter I. Those who prefer to assume the facts of Chapter I without comment should begin with Chapter II, which is a review of elementary algebraic operations.

larly, corresponding to any fraction $\frac{a}{b}$ (a and b integers), there can be constructed a point on $X'X$ such that the fraction denotes the distance and direction of the point from O. In fact, we assume that by means of this scale we are able to represent conveniently

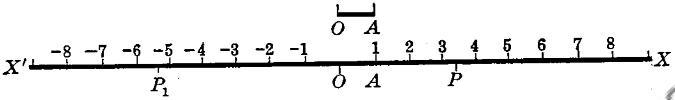

FIG. 1

all real numbers, and we can say, to any point P on the line, there corresponds a number * which indicates the distance and direction of P from O, and conversely, we assume that to every real number there corresponds a point of this line.

In addition to the real numbers, we shall find it desirable to deal with so-called "imaginary" and complex numbers. A graphical representation is given for these numbers in Art. 96.

ORAL EXERCISES

1. What numbers are represented by the following points?

(a) The point midway between 4 and 5.

(b) Points of trisection of the interval -3 to -4.

(c) Points of quadrisection of the interval 5 to 6.

2. State in words the position of points which represent $\frac{1}{2}, \frac{4}{3}, -\frac{3}{4}, -2, \pi$.

3. Suppose the scale of Fig. 1 represents the scale of a Fahrenheit thermometer; estimate the reading when the end of the mercury column stands at P_1.† At P. At point midway between O and P. At point midway between A and P.

4. A square piece of paper of side 2 is laid on Fig. 1 so that one corner of the square is at O and the diagonal lies on the line OX. The unit of measurement of the square is the same as the unit in Fig. 1. What number is represented by the point at the other corner of the square which lies on OX?

5. A circle of radius equal to two of the units in Fig. 1 rolls to the right along the line in Fig. 1, beginning at O. What number is represented by the point at which the circle touches the line after one complete turn?

3. Greater and less. The terms **greater than** and **less than** which are common to everyday life, when used in the technical

---

* For a more complete discussion, see Fine's *Number System of Algebra*, Second edition, p. 41.

† P_1 is read "P sub one." See list of signs and symbols at the end of the table of contents.

sense of algebra, are easily misunderstood. For this reason we point out their geometrical significance. The real number A is said to be greater than the real number B (written $A > B$) if the point representing A falls to the right of the point representing B. The number A is said to be less than the number B (written $A < B$) if the point representing A falls to the left of that representing B.

Exercise. Arrange the following numbers in ascending order of magnitude:

$1, \quad \sqrt{2}, \quad -4, \quad -\sqrt{3}, \quad 3\sqrt{3}, \quad -10, \quad -21, \quad \frac{1}{2}.$

4. Definitions and assumptions. Operations with numbers in arithmetic suggest certain definitions and rules for algebra. The student probably has performed algebraic operations according to rules thus suggested by arithmetic without being conscious of the assumptions which underlie these processes. We may now proceed to a formal statement of assumptions made at the outset in this algebra.

In algebra, a letter is used to represent a number. The **value** of the letter is the number which it represents. In the following, let a, b, c represent any numbers.

The fundamental * operations of **addition** and **multiplication** of numbers are subject to the following laws I–IX:

I. *The sum † of two numbers is a uniquely determined number.*

That is, given a and b, there is one and only one number x such that $a + b = x$.

II. *Addition is commutative.*

That is, $$a + b = b + a.$$

Illustrations: $3 + 5 = 5 + 3$, $4x + 5x = 5x + 4x$.

III. *Addition is associative.*

That is, $a + b + c = (a + b) + c = a + (b + c)$.

Illustrations: $10 + 3 + 7 = (10 + 3) + 7 = 10 + (3 + 7)$.

IV. *If equal numbers be added to equal numbers, the sums are equal numbers.*

That is, if $$a = b,$$
and $$c = d,$$
then $$a + c = b + d.$$

* The operations are fundamental in that no attempt is made to define them. The "laws" are in the nature of assumptions since no attempt is made to prove them.

† The **sum** is the result of adding.

V. *The product * of any two numbers is a uniquely determined number.*

That is, given a and b, there is one and only one number y such that $ab = y$. In this case, a and b are said to be **factors** of y.

VI. *Multiplication is commutative.*

That is, $$ab = ba.$$

VII. *Multiplication is associative.*

That is, $$(ab)c = a(bc).$$

VIII. *Multiplication is distributive with respect to addition.*

That is, $$a(b + c) = ab + ac.$$

Illustration: $4(x + 2y + 4z) = 4x + 8y + 16z.$

IX. *If equal numbers be multiplied by equal numbers, the products are equal numbers.*

That is, if $a = b$,
and $c = d$,
then $ac = bd$.

The following laws X and XI lead us to definitions of subtraction and division, and enable us to give meanings to the symbols $0, -a, \frac{a}{b}, 1$, and $\frac{1}{b}$.

X. *Given a and b, there is one and only one number x, such that $x + b = a$.*

Subtraction is the process of finding the number x in $x + b = a$. In other words, to **subtract** b from a is to find a number x, called the **remainder,** such that the sum of x and b is a.

By X, the number x in $x + b = a$ exists when $a = b$. In this case, the number x is called **zero,** and is written 0.

In symbols,
$$0 + b = b. \tag{1}$$

From X and the definition of 0, there exists a number x, such that
$$x + b = 0.$$

In this case, x and b are said to be negatives of each other, and x may be replaced by $(-b)$.

* The **product** is the result of multiplying.

If b is a positive number, x is a **negative** number.

In symbols,
$$(-b) + b = 0 \tag{2}$$
gives a definition of $(-b)$.

XI. *Given a and b ($b \neq 0$ *), there is one and only one number x, which satisfies $bx = a$.*

Division is the process of finding the number x in $bx = a$. In other words, to **divide** a by b is to find a number x, called the **quotient** of a by b, such that b multiplied by x gives a.

This quotient is often written $\frac{a}{b}$, and, when thus written, is called a **fraction.**

In dividing a by b, the number a is called the **dividend** and b the **divisor** just as in arithmetic. Likewise, in the fraction $\frac{a}{b}$, a is called the **numerator** and b the **denominator.**

By XI, the number x in
$$bx = a \quad (b \neq 0)$$
exists when $b = a$, so that $ax = a$. In this case, the number x is called **unity** and is written 1, that is,
$$\frac{a}{a} = 1. \tag{3}$$

Further, by XI, and the definition of 1, there exists a number x, which satisfies $bx = 1$ ($b \neq 0$). This value of x is called the **reciprocal** of b and is written $\frac{1}{b}$.

It should be noted that, by means of XI, a meaning is given to unity and to the reciprocal of any number other than zero in a manner analogous to that by which a meaning is given to zero and to the negative by means of X.

The above propositions I–XI are stated for two and three numbers for the sake of simplicity. It can be proved from the given assumptions that these propositions hold when three or more numbers are concerned in the process of addition or multiplication.

It is not to be inferred that these propositions I–XI are entirely independent of one another, but rather that they constitute convenient assumptions for the purposes of this course in algebra.

* The sign $\neq$ stands for "*is not equal to.*"

Although further assumptions are made later in the course, the principles I–XI enable us to prove many important propositions of algebra, and the wide application of algebra depends upon the fact that many changes in the physical world take place in accordance with these laws.

As shown in the following exercises, the operations of algebra are generalizations of the operations of elementary arithmetic.

EXERCISES

1. Show that $2 \times 5,\ 3 \times 6,\ 5 \times 8,\ -1 \times 2,\ 0 \times 3,\ 11 \times 14,\ \frac{1}{2} \times 3\frac{1}{2}$ are all special cases of $x(x + 3)$ and of $(x - 4)(x - 1)$.

2. Write an algebraic expression of which the products 3×7, 4×8, 6×10, 12×16 are special cases.

3. Show that $4 \times 5 \times 8, 6 \times 7 \times 10, 0 \times 1 \times 4, \frac{1}{2} \times \frac{3}{2} \times \frac{9}{2}, -\frac{1}{2} \times \frac{1}{2} \times \frac{7}{2}$ are special cases of $x(x + 1)(x + 4)$ and of $(x - 3)(x - 2)(x + 1)$.

4. Show that $\frac{-5}{5} = -1,\ \frac{0}{6} = 0,\ \frac{7}{7} = 1,\ \frac{16}{8} = 2,\ \frac{27}{9} = 3$ are special cases of $\frac{x^2 - 9}{x + 3} = x - 3$.

In the course of operations with the numbers of algebra, the important question arises: Can any two given numbers be added, subtracted, multiplied, or divided? Our assumptions state that the number system of algebra is such that this question can be answered in the affirmative **except in the case of division by zero.** *Division by zero is excluded from algebraic operations.*

EXERCISES

1. Can any two given numbers be added, subtracted, multiplied, or divided if the number system consists of positive integers only? Illustrate your answer.

2. Can any two given numbers be added, subtracted, multiplied, or divided if the number system consists of positive integers and quotients of positive integers only? Illustrate your answer.

3. Where is the flaw in the following?

Let	$x = a \quad (x \neq 0).$	(1)
Multiply both sides by x,	$x^2 = ax.$	(2)
Subtract a^2 from both sides,	$x^2 - a^2 = ax - a^2.$	(3)
Factor,	$(x - a)(x + a) = a(x - a).$	(4)
Divide both sides by $x - a$,	$x + a = a.$	(5)
But, by (1),	$x = a.$	(6)
By (5) and (6),	$2a = a.$	(7)
Hence,	$2 = 1.$	

5. Derived properties of the numbers of algebra. From the foregoing definitions and assumptions, the following propositions can be proved. We shall present in detail the proofs of only a few.

I. *Adding a negative number* $(-a)$ *is equivalent to subtracting a positive number* a.

That is, $b + (-a) = b - a.$

To prove this, let $b + (-a) = x.$ (I, Art. 4.) (1)

$b + (-a) + a = x + a.$ (IV, Art. 4.) (2)

But $(-a) + a = 0.$ (Eq. (2), page 5.) (3)

From (2) and (3) $b + 0 = x + a.$ (4)

But $b + 0 = b.$ (Eq. (1), page 4.) (5)

From (4) and (5) $b = x + a.$ (6)

$b - a = x.$ (Def. of subtraction.) (7)

That is, $b + (-a) = b - a.$

Illustration: $5 + (-4) = 5 - 4 = 1.$

II. *Subtracting a negative number* $(-a)$ *is equivalent to adding the positive number* a.

That is, $b - (-a) = b + a.$

Illustration: $6 - (-2) = 6 + 2 = 8.$

III. *The product of two numbers is* 0 *when and only when at least one of the numbers is* 0.

COROLLARY. *The quotient* $\frac{0}{a}$ *is equal to* 0 *when* a *is any number different from* 0.

Illustrations: $100 \times 0 = 0,\ 0 \times 25 = 0,\ \frac{0}{27} = 0.$

IV. *The product of a number* a *by a number* $(-b)$ is $-ab$.

To prove this, let

$a(-b) = x.$ (x exists by V, Art. 4.) (1)

Then $a(-b) + ab = x + ab.$ (IV, Art. 4.) (2)

$a[(-b) + b] = x + ab.$ (VIII, Art. 4.) (3)

$a \cdot 0 = x + ab.$* (Eq. (2), page 5.) (4)

$0 = x + ab.$ (III, Art. 5.) (5)

$-ab = x.$ (Definition of negative.) (6)

From (1) and (6),

$$a(-b) = -ab.$$

Illustration: $3(-4) = -3 \cdot 4 = -12.$

* The $\cdot$ is a sign of multiplication, thus, $5 \cdot 6 = 30.$

V. *The product of* $(-a)$ *by* $(-b)$ *is* ab.

To prove this, let

$$(-a)(-b) = x. \qquad (1)$$

$$(-a)(-b) + a(-b) = x - ab. \qquad \text{(IV, Art. 4; IV, Art. 5.)} \qquad (2)$$

$$(-b)[(-a) + a] = x - ab. \qquad \text{(VIII, Art. 4.)} \qquad (3)$$

$$-b \cdot 0 = x - ab. \qquad \text{(Definition of zero.)} \qquad (4)$$

$$0 = x - ab. \qquad \text{(Why?)} \qquad (5)$$

$$ab = x. \qquad \text{(Why?)} \qquad (6)$$

From (1) and (6), $(-a)(-b) = ab.$

Illustration: $(-5)(-7) = 35.$

The statement that in multiplication *like signs give plus and unlike signs give minus* includes IV and V.

VI. *The quotient of two numbers is positive if the signs of the dividend and divisor are alike; negative if they are unlike.*

VII. *A single parenthesis may be removed when it is preceded by a positive sign without changing the signs of the terms within it.*

VIII. *A single parenthesis may be removed when preceded by a negative sign if the sign of each term within it is changed.*

That is, $-(a + b - c + d - e) = -a - b + c - d + e.$

Illustration: $10 - (5 - 2) = 10 - 5 + 2 = 7.$

IX. *The value of a fraction is not changed by multiplying or dividing both the numerator and denominator by the same number, not zero.*

That is, $$\frac{a}{b} = \frac{ax}{bx} \quad (x \neq 0)$$

Illustrations: $$\frac{3}{5} = \frac{3 \cdot 4}{5 \cdot 4} = \frac{12}{20},$$

$$\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}.$$

X. *Changing the sign of either the numerator or the denominator of a fraction is equivalent to changing the sign of the fraction.*

That is, $$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}.$$

Illustration: $$\frac{-5}{8} = -\frac{5}{8} = \frac{5}{-8}.$$

XI. *Adding two fractions having a common denominator gives a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator.*

That is, $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Likewise, $$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Illustrations: $$\frac{7}{5} + \frac{3}{5} = \frac{10}{5}, \quad \frac{7}{5} - \frac{3}{5} = \frac{4}{5}.$$

XII. *The sum and the difference of two fractions are expressed by*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \textit{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}, \textit{ respectively.}$$

We can reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator, since, by IX,

$$\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$

By XI, we can complete the process.

Illustration: $$\frac{3}{4} - \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 3} - \frac{2 \cdot 4}{3 \cdot 4} = \frac{9-8}{12} = \frac{1}{12}.$$

XIII. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

That is, $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

XIV. *To divide one fraction by another, invert the latter and then multiply one by the other.*

That is, $$\frac{a}{b} \div \frac{c}{d} \text{ or } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

Illustration: $$\frac{3}{7} \div \frac{4}{5} = \frac{3}{7} \cdot \frac{5}{4} = \frac{15}{28}.$$

Propositions III, XI, XIII, stated for two numbers in each case, are readily extended to three or more numbers.

ORAL EXERCISES

1. What is the product of 0 and any given number?
2. What is the quotient when 0 is divided by any number other than 0?
3. Explain why division by 0 is excluded.

Carry out the indicated operations in each of the following:

4. $5 - (-3)$.
5. $6x - (-3x)$.
6. $5x - (1 - 2x)$.
7. $-(3 - 6b + x)$.
8. $(-3)(-4)$.
9. $(-5x)(-4y)$.
10. $(-5x)(4z)$.
11. $\frac{-10}{-2}$.
12. $\frac{32}{-4}$.
13. $\frac{-6x}{3x}$.
14. $\frac{-6x}{-3x}$.
15. $\frac{1}{3} + \frac{1}{2}$.
16. $\frac{x}{3} + \frac{x}{2}$.
17. $\frac{x}{2} - \frac{x}{3}$.
18. $\frac{2}{3} \cdot \frac{3}{4}$.
19. $\frac{2x}{3} + \frac{3x}{4}$.
20. $\frac{2x}{3} - \frac{3x}{4}$.
21. $\frac{3}{4} \div \frac{3}{2}$.
22. $\frac{3}{2} \div \frac{15}{4}$.
23. $6 \div \frac{2}{3}$.
24. $6a \div \frac{3a}{6}$.
25. $5x \div \frac{15x}{8}$.
26. $4a \div \frac{2a}{5}$.

CHAPTER II

A REVIEW OF CERTAIN ELEMENTARY OPERATIONS OF PRE-COLLEGE ALGEBRA

6. Introduction. In algebra as in arithmetic there are four fundamental operations — addition, subtraction, multiplication, and division. In this chapter our main interest is in a review of these operations by means of their repeated use, especially in the important process of reducing certain algebraic expressions to simpler forms.

7. Algebraic expressions. In algebra, an **expression** is a symbol or combination of symbols that represents a number. Thus,

$$x^2 + y^2 - 25$$

and
$$\tfrac{1}{2}gt^2 + vt$$

are expressions, if x, y, g, t, v represent numbers. If $x = 4$ and $y = 6$, the first takes the value 27. If $g = 32.2$, $t = 10$, and $v = 5$, the second has the value 1660. For different values of the letters, an expression represents in general different numbers. On the other hand, the same numbers may be represented by many different expressions. For example, $x^2 - 4$ and $(x - 2)(x + 2)$ represent the same number. Expressions which are equal for all values of the symbols for which the expressions are defined * are called **identical expressions.** The statement that two identical expressions are equal is called an **identity.** Thus, $x^2 - 4$ and $(x - 2)(x + 2)$ represent the same number no matter what value is assigned to x, and the statement

$$x^2 - 4 = (x - 2)(x + 2)$$

is an identity.

Two expressions may be equal without being identical. Thus, for $x = 2$ or -5, the expressions $x^2 + 2x - 1$ and $9 - x$ are equal, but they are equal for no other values of x, and hence are not iden-

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* This statement implies that we may not assign values to the letters which make the members meaningless. Thus, $\dfrac{1}{1-x} = 1 + \dfrac{x}{1-x}$ is excluded when $x = 1$.

tical. Frequently in problems with which we shall deal, the work is made easier by replacing expressions by identical but simpler expressions.

Exercise. Which of the following are identical expressions?

$$3x, \quad (x+1)(x-1), \quad \frac{3x^2-3x}{x-1}, \quad 4x-x, \quad x^2-1, \quad \frac{9x^2}{3x}.$$

8. Use of parentheses. In a sum of a number of parts, each part with the sign that precedes it is called a **term.** Thus, in $(3a - 2x + 9)$ the terms are $3a$, $-2x$, and 9.

In order to group terms together, we use parentheses. It should be remembered that parentheses may be removed with or without change of sign of each term included, according as the sign $-$ or $+$ precedes the parentheses.

Thus
$$a - (b - c) = a - b + c,$$
$$a + (b - c) = a + b - c.$$

Expressions often occur with more than one pair of parentheses. When one pair occurs within another pair, other symbols besides ( ) are used as follows: [ ] called **brackets,** { } called **braces,** and $\overline{}$ called the **vinculum.** All parentheses may be removed by first removing the innermost pair according to the rule for a single pair; next, the innermost pair of all that remain, and so on. Thus,

$$\begin{aligned} & a - \{3x - 4[x - (y - 5) - (2y + 6)]\} \\ = {} & a - \{3x - 4[x - y + 5 - 2y - 6]\} \\ = {} & a - \{3x - 4x + 12y + 4\} \\ = {} & a + x - 12y - 4. \end{aligned}$$

EXERCISES

Remove the parentheses and other signs of grouping from the expressions in the first eight exercises.

1. $4x - 5 + (y - 2x) + 3(y - x)$.

2. $-[2x + (3 - 4x)]$.

3. $-(t - 2) - [2t - (t - 3)]$.

4. $2x - (3x - a) - (a - 2x)$.

5. $7s - [-8t - (10s - 11t)]$.

6. $-\{(3a - 2a) - (a + 2)\}$.

7. $6m - \{2m - [(m + 2m) - (2m - \overline{m + 1})]\}$.

8. $-p + q - \{r - [(p - q + r) - (-\overline{p + q} - r)]\}$.

In each of the following four expressions, inclose the last three terms, first in parentheses preceded by a plus sign, then in parentheses preceded by a minus sign.

9. $7 + 3t + 5y - 7.$

10. $a - 2b + 3c - 4d.$

11. $5 + 2x + 4 - 5y.$

12. $a + b + c - x - y - z.$

13. Find the value of the expression

$$5x - [3x + (-4x - \overline{1 - x}) + 1 - (-3 - x)]$$

when $x = 2$.

14. Find the value of the expression

$$[2(2 + y) - y(2 - y)] - [2(2 - y) + y(2 + y)]$$

when $y = 7$.

Fill out the parentheses in the following:

15. $4a - 5b + 2 - (\quad\quad) = a + b.$

16. $6m + 4n - 3 - 5m + n - (\quad\quad) = 1.$

9. Factoring. Many algebraic expressions are readily factored when certain type products are recognized. The student will probably recall the following from his study of high school algebra.

a. Common monomial factor.

$$ax + ay = a(x + y).$$

Illustration: $2am - 4a^2 = 2a(m - 2a).$

b. Difference of two squares.

$$a^2 - b^2 = (a + b)(a - b).$$

Illustration: $81x^2 - 1 = (9x + 1)(9x - 1).$

c. Perfect trinomial square.

$$a^2 \pm 2ab + b^2 = (a \pm b)^2.$$

Illustration: $4x^2 - 12xy + 9y^2 = (2x - 3y)^2.$

d. Trinomial of the form

$$x^2 + (a + b)x + ab = (x + a)(x + b).$$

Illustration: $x^2 + 7x + 10 = (x + 5)(x + 2).$

e. Trinomial of the form

$$ax^2 + bx + c.$$

Certain expressions of this form can be factored by inspection.

Illustration: $2x^2 + 5x - 3 = (2x - 1)(x + 3).$

f. Sum of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Illustration:

$$125x^6 + 8y^{15} = (5x^2)^3 + (2y^5)^3 = (5x^2 + 2y^5)(25x^4 - 10x^2y^5 + 4y^{10}).$$

g. Difference of two cubes.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Illustration: $m^3y^3 - 27n^9 = (my - 3n^3)(m^2y^2 + 3mn^3y + 9n^6)$.

h. Factors found by grouping.

$$ax + ay + bx + by = (a + b)(x + y).$$

Illustration:

$$2ax + 4bx - 3ay - 6by = 2x(a + 2b) - 3y(a + 2b) = (2x - 3y)(a + 2b).$$

i. Cube of a binomial. By performing the multiplication, we find that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Illustration:
$$\begin{aligned} & 8x^3 + 36x^2y + 54xy^2 + 27y^3 \\ &= (2x)^3 + 3 \cdot (2x)^2 \cdot (3y) + 3 \cdot (2x) \cdot (3y)^2 + (3y)^3 \\ &= (2x + 3y)^3. \end{aligned}$$

The student should find the cube of $a - b$.

j. Expressions that can be reduced to the form of the difference of two squares.

Illustration 1. Factor $4x^4 + y^4$.

By the addition and subtraction of $4x^2y^2$ we have

$$\begin{aligned} 4x^4 + y^4 &= 4x^4 + 4x^2y^2 + y^4 - 4x^2y^2 \\ &= (2x^2 + y^2)^2 - 4x^2y^2 \\ &= (2x^2 + y^2 - 2xy)(2x^2 + y^2 + 2xy). \end{aligned}$$

Illustration 2. Factor $a^4 + 3a^2b^2 + 36b^4$. Find two factors only.

Since the expression $a^4 + 3a^2b^2 + 36b^4$ is a trinomial in which a^4 and $36b^4$ are perfect squares, we compare the middle term, $3a^2b^2$, with the middle term of $(a^2 + 6b^2)^2 = a^4 + 12a^2b^2 + 36b^4$. If $9a^2b^2$ be subtracted from this expression we have $a^4 + 3a^2b^2 + 36b^4$.

Or we may write

$$\begin{aligned} a^4 + 3a^2b^2 + 36b^4 &= a^4 + 12a^2b^2 + 36b^4 - 9a^2b^2 \\ &= (a^2 + 6b^2)^2 - (3ab)^2 \end{aligned}$$

which is the difference of two squares and can be factored as

$$(a^2 + 6b^2 + 3ab)(a^2 + 6b^2 - 3ab).$$

ORAL EXERCISES

Practice in Factoring Expressions Which Reduce to Important Type Products

Factor the following:

1. $3ax + 6a^2$.
2. $4a^2 - y^2$.
3. $36x^2 - a^2$.
4. $9x^2 + 6bx + b^2$.
5. $a^2 + 4ab + 4b^2$.
6. $x^2 - 4xy + 4y^2$.
7. $y(x - 8) - 5(x - 8)$.
8. $(x - y)^2 - ax + ay$.
9. $15(x^2 - xy) - 10(xy - y^2)$.
10. $25y^2 - 90y + 81$.
11. $x^3 + 3x^2y + 3xy^2 + y^3$.
12. $x^3 - 3x^2y + 3xy^2 - y^3$.
13. $x^3 + 8y^3$.
14. $x^3 - 8y^3$.
15. $4x^2 - 12x + 9$.
16. $x^2 + 17x + 72$.

WRITTEN EXERCISES

Practice in Factoring More Complicated Expressions

Factor:

1. $4x^4 - 9y^4$.
2. $(a + b)^2 - (a - b)^2$.
3. $16x^4 + 4y^4$. Find two factors.
4. $s^2 - 12s + 35$.
5. $16x^2 - \frac{1}{9}a^2$.
6. $4x^2 + 7xy + 3y^2$.
7. $-2x^2 + 15 + x$.
8. $25x^2y^2 - 30xy + 9$.
9. $15ay^2 - 11ay + 2a$. Find three factors.
10. $(x + 3)^2 + 2(x + 3) + 1$.
11. $y^2 - (h + k)^2$.
12. $x^2 + 4xy + 4y^2 - a^2 - 2ab - b^2$.
13. $9a^2 - 6ab + b^2 - 25x^2 - y^2 - 10xy$.
14. $4x^4 - 20x^2y^2 + 25y^4$.
15. $x^4 + x^2 + 1$. *Hint:* Add and subtract x^2.
16. $9x^4 + 11x^2 + 4$.
17. $x^4 + 4y^4$.
18. $h^3 + y^3$.
19. $27 - s^3$.
20. $u^3 + 1000$.
21. $8x^3 - 12x^2y + 6xy^2 - y^3$.
22. $x^6 - y^6$. Find four factors.
23. $x^5 - y^5$. Find two factors.
24. $(a - b)^4 - (a + b)^4$.
25. $(x + y)^3 - (x - y)^3$.
26. $(x + y)^3 + (x - y)^3$.
27. $(x + y)^4 - (x - y)^4$.
28. $a^3 + b^3 + a + b$.
29. $a^2 + 2ab + b^2 - 4a - 4b + 4$.
30. $21x^3 - 14x^2y - 56xy^2$.
31. $2(x - 3)^3(2x + 1) - 3(2x + 1)^2(x - 3)^2$.
32. $(a - 2b + c)^2 - (2b - c - 3a)^2$.
33. $ac + d - c - ad$.
34. $4ay - 4a^2 + x^2 - y^2$.

10. Fractions. In high school algebra, the student has learned to add, subtract, multiply and divide simple fractions. But in order that he may regain his facility for carrying out these fundamental operations he should recall the following important principle.

The value of a fraction is not changed by multiplying, or dividing both numerator and denominator by the same number, not zero. (See IX, Art. 5.)

Illustration 1. Reduction of fractions to lowest terms.

Thus, $\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}$; $\frac{15a^2x^2y}{20a^3xy} = \frac{3x}{4a}$; $\frac{4x - 4y}{x^2 - y^2} = \frac{4}{x+y}$.

Illustration 2. Reduction of two or more fractions to a common denominator in addition or subtraction.

Thus,

$$\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12};\quad \frac{1}{a-b} + \frac{1}{a+b} = \frac{a+b}{a^2-b^2} + \frac{a-b}{a^2-b^2} = \frac{2a}{a^2-b^2}.$$

In this connection it is well to recall the definition of a **reciprocal.** (See XI, Art. 4.)

DEFINITION. The reciprocal of a number, say of N, is the fraction $\frac{1}{N}$. Thus, the reciprocal of 3 is $\frac{1}{3}$; of $\frac{a}{b}$ is $\frac{b}{a}$.

ORAL EXERCISES

Give the reciprocal of each of the following:

1. $\frac{3}{2}$. **2.** 0.5. **3.** 1. **4.** -1.

5. What number has no reciprocal?

6. Change $\frac{2}{3}$ to a fraction whose denominator is 24.

Reduce to lowest terms:

7. $\frac{20}{45}$. **9.** $\frac{a^2+ab}{a^2}$. **11.** $\frac{x}{x-x^2}$.

8. $\frac{25a^3b^3c}{5a^2b^2c}$. **10.** $\frac{y-x}{(y-x)^2}$. **12.** $\frac{(a+b)(x^2-y^2)}{(a+b)(x-y)}$.

WRITTEN EXERCISES

Practice in the Simplification of Fractions

Reduce to lowest terms:

1. $\frac{(a-b)(x-y)}{(a-b)(x^2-y^2)}$. **4.** $\frac{x(2a-b)}{5(b-2a)}$.

2. $\frac{a^2-b^2}{a^3-b^3}$. **5.** $\frac{-2a-2b}{(a+b)^2}$.

3. $\frac{x-y}{x^2-y^2+x-y}$. **6.** $\frac{8x^3-y^3}{y^2-4xy+4x^2}$.

Combine into a single fraction:

7. $\frac{7}{10} + \frac{1}{2} - \frac{4}{5}$.

8. $\frac{2b}{a^2 - b^2} + \frac{2}{a + b}$.

9. $\frac{3}{x + y} - \frac{3x}{x^2 + 2xy + y^2}$.

10. $\frac{x - 9}{7(x - 2)} + \frac{4}{x^2 - 4}$.

11. $\frac{s + 5}{3(s - 3)} - \frac{3s + 7}{s^2 - 9}$.

12. $5 + \frac{3t}{t - 2} + 4t$.

13. $x + 7 - \frac{x^2}{x - 7}$.

14. $\frac{5}{x^2-9} + \frac{2-x}{x^2+x-6} + \frac{x - 4}{x^2 - 7x + 12}$.

Simplify:

15. $\frac{2a - 2b}{a + b} \cdot \frac{b}{a - b}$.

16. $\frac{a - b}{a + b} \cdot \frac{b}{a^2 - b^2}$.

17. $\frac{x(x^2 - y^2)}{x + y} \cdot \frac{y(x^2 + y^2)}{x - y}$.

18. $\frac{x^2}{x^2 + y^2} \cdot \frac{x - y}{x^2 + xy} \cdot \frac{x^4 - y^4}{(x - y)^2}$.

19. $\frac{a^2 - b^2}{(a - b)^2} \div \frac{a^2 + ab}{a - b}$.

20. $\frac{x^3 - 4x}{x^2 + 5x + 6} \div \frac{x^2 - 3x + 2}{x^2 + 2x - 3}$.

21. $\frac{a^4 - b^4}{(a - b)^2} \div \frac{a^2 + b^2}{a^2 - ab}$.

22. $\frac{x^2 - 16}{4x^2} \div \frac{x^2 + 4x}{12}$.

11. Complex fractions. A **complex fraction** is one which has a fraction in the numerator or denominator or in both numerator and denominator. The rules for the simplification of arithmetical fractions apply to algebraic fractions no matter how complicated the numerator or denominator may be.

The main principle is that the value of a fraction is not changed by multiplying numerator and denominator by the same number.

As illustrated by the following examples and exercises on complex fractions, a simplification is often brought about if we select for the number by which to multiply, the lowest common denominator of the fractions which are in the numerator and denominator of the complex fraction.

Example 1. Simplify the complex fraction $\frac{\frac{2}{3} - \frac{1}{2}}{\frac{3}{4} - \frac{5}{12}}$.

Solution: Since 12 is the lowest common denominator of the fractions in the numerator and denominator of the given fraction, we multiply the numerator and denominator of the given fraction by 12. We have

$$\frac{\frac{2}{3} - \frac{1}{2}}{\frac{3}{4} - \frac{5}{12}} = \frac{12(\frac{2}{3} - \frac{1}{2})}{12(\frac{3}{4} - \frac{5}{12})} = \frac{8 - 6}{9 - 5} = \frac{2}{4} = \frac{1}{2}.$$

Example 2. $$\frac{\frac{1}{b} + \frac{1}{a + b}}{\frac{1}{a} - \frac{1}{a - b}} = \frac{ab(a^2 - b^2)\left[\frac{1}{b} + \frac{1}{a + b}\right]}{ab(a^2 - b^2)\left[\frac{1}{a} - \frac{1}{a - b}\right]}$$

$$= \frac{a(a^2 - b^2) + ab(a - b)}{b(a^2 - b^2) - ab(a + b)} = -\frac{a(a^2 - 2b^2 + ab)}{b^2(a + b)}.$$

EXERCISES

Simplify the following fractions:

1. $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{5}{6}}$.

2. $\dfrac{2-\frac{2}{3}}{1+\frac{1}{6}}$.

3. $\dfrac{a+\dfrac{1}{b}}{b+\dfrac{1}{a}}$.

4. $\dfrac{\dfrac{a}{b}+\dfrac{x}{y}}{\dfrac{a}{b}-\dfrac{x}{y}}$.

5. $\dfrac{x+\dfrac{1}{x}+1}{x^2-\dfrac{1}{x}}$.

6. $\dfrac{5\frac{1}{3}}{16-2\frac{2}{3}}$.

7. $\dfrac{\dfrac{2}{x}-\dfrac{3}{y}}{\dfrac{5}{x}-\dfrac{6}{y}}$.

8. $\dfrac{\dfrac{m^2-1}{2n^2}}{\dfrac{m+1}{n}}$.

9. $\dfrac{4+\dfrac{a}{b-a}}{1+\dfrac{2a}{b-a}}$.

10. $\dfrac{\dfrac{a}{a-1}-1}{1+\dfrac{a}{1-a}}$.

11. $\dfrac{9y^2-\dfrac{4}{x^2}}{3x-\dfrac{2}{y}}$.

12. $\dfrac{x-\dfrac{1}{y}}{y-\dfrac{1}{x}}$.

13. $\dfrac{100-\dfrac{1}{a^2b^2}}{2a-\dfrac{1}{5b}}$.

14. $\dfrac{\dfrac{c}{ab}}{\dfrac{b}{a}+\dfrac{c}{b}}$.

15. $\dfrac{3ab+4ac}{4+\dfrac{3b}{c}}$.

16. $\dfrac{1+\dfrac{1}{x^2}+\dfrac{1}{x^4}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}$.

17. $\dfrac{4x^2-\dfrac{9y^2}{x^2}}{2x^2y+3y^2}$.

18. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$.

19. $(6a^2-5ab-21b^2)\div\left(4-\dfrac{a+7b}{a}\right)$.

20. $\dfrac{1+\dfrac{1}{x}}{\dfrac{1}{x}}\div\dfrac{1+x^2}{\dfrac{1}{x^2}}$.

21. $\dfrac{\dfrac{4a^2+3a}{1+\dfrac{a+2}{3a-2}}-3a^2}{2-\dfrac{7a+4}{4a-1}}$.

22. $\dfrac{\dfrac{m}{m-1}+\dfrac{m}{m+1}}{\dfrac{m}{m-1}-\dfrac{m}{m+1}}$.

23. $1-\dfrac{1}{2-\dfrac{1}{3-\dfrac{1}{4+a}}}$.

24. $\dfrac{\left(\dfrac{a^2+b^2}{b}-a\right)\div\left(\dfrac{1}{b^3}+\dfrac{1}{a^3}\right)}{\left(\dfrac{1}{b}-\dfrac{1}{a}\right)\div\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)}$.

MISCELLANEOUS EXERCISES AND PROBLEMS

Remove symbols of grouping and simplify:

1. $7x - [-8y - (10x - 11y)]$. **2.** $b - \{3a - [a - \overline{2 - a} - 2] - 2\}$.

Inclose the last two terms in parentheses preceded by a plus sign; by a minus sign:

3. $x^2 + 4x - 16 + y^2 - 8y$.

4. $x^2 + 8x - y + 10$.

5. $3x^2 + 6x - 2y^2 + 8$.

Inclose the terms containing y in parentheses preceded by a plus sign; by a minus sign:

6. $xy - 2x + 4y$. **7.** $x^2y^2 - a^2x^2 + ay^2$. **8.** $x^3 + xy^2 + ax^2 - ay^2$.

Factor:

9. $x^3 + 27y^3$.

10. $ax^2 + 5ax - cx^2 - 5cx$.

11. $x^4 + x^2y^2 + y^4$.

12. $x^4 + 4$.

Simplify:

13. $\dfrac{\dfrac{m^2 - 4n^2}{mn - 2n^2}}{\dfrac{m^2 + 2mn}{m^2 + 4n^2}}$.

14. $\dfrac{\dfrac{1}{p - q} - \dfrac{1}{p - r}}{\dfrac{q^2}{p^2 - p(q + r) + qr}}$.

15. $\dfrac{\dfrac{1}{2x} + 2x + 2}{\dfrac{8x + 5}{2x^2} - 2}$.

16. $\dfrac{x^2 - x + 1 - \dfrac{1}{x + 1}}{1 - \dfrac{1}{x + 1}}$.

17. $\dfrac{\dfrac{1}{a} + \dfrac{1}{b}}{\dfrac{a^2 - b^2}{ab}}$.

18. $\dfrac{\dfrac{1}{x + y} - \dfrac{x}{x^2 - y^2} + \dfrac{1}{x - y}}{\dfrac{x + y}{x - y} - \dfrac{x - y}{x + y}}$.

19. A tells B to think of two numbers each less than 10, then to multiply one of the numbers by 5, add 4 to the product, multiply this sum by 2, and add the other number. B announces the final result of this computation and A immediately tells him what two numbers he thought of. A does this by mentally subtracting 8 from B's result. The number then obtained is a two-digit number, the digits being B's two numbers. Show the algebra underlying this trick.

Hint: Let x and y be the two numbers.

20. Make up another trick like that in problem 19.

Evaluation of Formulas Involving Fractions

21. The present value P at simple interest of a sum S due in n years at interest rate i per year is given by the formula

$$P = \frac{S}{1 + ni}.$$

Find P when $S = \$1000$, $n = \dfrac{1}{2}$, $i = .05$.

22. The speed v in feet per second of a projectile of weight w pounds, and diameter d inches is given by

$$\frac{1}{v} = \frac{1}{v_0} + \frac{td^2}{7000w}$$

where v_0* is muzzle speed with which the shell is projected, and t is the number of seconds after leaving the muzzle. Find v when $v_0 = 2750$, $t = 5$, $d = 14$, $w = 1200$.

23. $H = \dfrac{v}{825}\left(T - \dfrac{wv^2}{g}\right)$.

Find H when $T = 390$, $w = 0.7$, $g = 32.2$, and $v = 88$.

24. $T = \dfrac{\left(P + \dfrac{a}{v^2}\right)(v - b)}{R}$.

Find T when $a = 1322$, $b = 0.01969$, $R = 36.24$, $v = 1.590$, $P = 11100$.

25. The specific gravity S of a floating body is given by the expression

$$S = \frac{w_1}{w_1 + (w_2 - w_3)}$$

where w_1 is the weight of the body in air, w_2 is the weight of a sinker in water, and w_3 is the weight in water of the body with sinker attached.

Determine the specific gravity of a body when by physical measurements it is found that

$$w_1 = 17.36$$
$$w_2 = 193.7$$
$$w_3 = 186.8$$

26. One cubic centimeter of mercury at x degrees centigrade increases in volume when heated to y degrees by an amount given by the following formula:

$$\frac{\dfrac{A(y - x)}{100}}{1 + \dfrac{Ax}{100}},$$

where $A = 0.018$. Find the increase in volume when the temperature is raised from 11° to 127°.

27. To correct a barometer reading for temperature the following amount is subtracted from the reading:

$$B\frac{m(t - 32) - s(t - 62)}{1 + m(t - 32)},$$

where B is the barometer reading in inches, t the temperature in degrees Fahrenheit, $m = 0.00010$, $s = 0.00001$. What is the corrected reading of the barometer when the temperature is 86 and the barometer reads 30.15?

* A letter with a subscript, say a_r, is read, "a sub r."

28. Let P be the day of the month, q the number of the month in the year, counting January and February as the 13th and 14th months of the preceding year, N the year, and $n = \left[\frac{N}{100}\right] - \left[\frac{N}{400}\right] - 2$.

If
$$P + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - n$$
be divided by 7, the remainder will be the day of the week of a given date where Sunday counts as the first day. The expressions in brackets mean the largest integer contained in the inclosed number. Verify this formula for the present date.

29. The formula for the horsepower H.P. of an automobile engine is given by
$$\text{H.P.} = \frac{Planc}{(24)(33000)},$$
where P is the pressure in pounds per square inch, l is the length of stroke of the piston in inches, a is the area of the end of the piston in square inches, n is the number of revolutions of the flywheel per minute, c is the number of cylinders in the engine. How many H.P. are developed by a six-cylinder engine if P is 74, l is 5.5, a is 15.56, and n is 1250?

CHAPTER III

FUNCTIONS AND THEIR GRAPHS

12. Constants and variables. A **constant** is a symbol which represents the same number throughout a discussion. A **variable** is a symbol which may represent different numbers in the discussion or problem into which it enters. Many mathematical expressions contain both variables and constants. Except in certain geometrical and physical formulas it is customary to use the letters $a, b, c, \ldots$ from the beginning of the alphabet for constants and the letters $\ldots x, y, z$, from the end of the alphabet for variables.

Exercise. If A and B are two points in a plane and a point P moves in a circle about A as a center, which of the distances PA, PB is constant? which variable?

13. The function idea. Many problems in mathematics, physics, engineering, and chemistry involve two variables which are so related that, a value of one being given, the other can be found. The relation between the variables may be exhibited in various ways. Sometimes the values of the variables are arranged in the form of a table. For example, a life insurance agent refers to a table to find the premium corresponding to a given age. Here the two variables are "premium" and "age."

In algebra one variable may be connected with another in an equation or one variable may be an algebraic expression containing the other. In the equation $3x - 5y = 4$, if a value be given to x the corresponding value of y can be found. Thus if $x = 0$, $y = -\frac{4}{5}$; if $x = 1$, $y = -\frac{1}{5}$, and so on.

In evaluating the expression $x^2 + x + 1$, we find that $x^2 + x + 1 = 1$ when $x = 0$, $x^2 + x + 1 = 3$ when $x = 1$, and so on. Fixing the value of x in the first illustration fixes the value of y; in the second illustration fixing the value of x fixes the value of $x^2 + x + 1$.

DEFINITION OF A FUNCTION. If two variables are so related that when a value of one is given, a corresponding value of the other is determined, the second variable is called a **function** of the first.

Thus in the equation $3x - 5y = 4$, y is a function of x. The expression $x^2 + x + 1$, and in general any expression containing x, is a function of x. We may therefore and shall speak of a "function of x" instead of "an expression involving the variable x."

14. Functional notation. The symbols $f(x)$, $g(x)$, $\phi(x)$, . . . are used to represent functions of the variable x. For example, let $f(x)$ represent $x^4 + 3x^2 - 2x + 10$. The symbol $f(x)$ is read "the f function of x." Similarly "$\phi(y)$" is read "the ϕ function of y," (pronounced "phi function of y").

To illustrate further, suppose that in a discussion

$$f(x) = 3x^2 - 2x + 1,$$

then $$f(2) = 3 \cdot 2^2 - 2 \cdot 2 + 1 = 9,$$

and $$f(a) = 3a^2 - 2a + 1.$$

Similarly, if $$g(x) = x^2 + 4x,$$

then $$g(b) = b^2 + 4b,$$

and $$g(3) = 3^2 + 4 \cdot 3 = 21,$$

$$g(-2) = (-2)^2 + 4(-2) = -4.$$

These illustrations bring out an important point in the functional notation, namely: If the same functional symbol, say $g(\;)$, be used more than once in a discussion, it stands in each case for the same operation or set of operations on the number or expression contained in the parentheses of the functional symbol.

ORAL EXERCISES

Express in words exercises 1–4.

1. $A = f(a)$. **2.** $s = F(t)$. **3.** $y = \phi(x)$. **4.** $w = H(u)$.

5. If $f(x) = 5x - 3$, find $f(1), f(2), f(-2), f(0)$.

6. If $F(x) = x^2 + 9x - 3$, find $F(1)$, $F(0)$, $F\left(\frac{1}{2}\right)$, $F(a)$, $F(-a)$.

WRITTEN EXERCISES

1. If $\phi(t) = t^2 - t - 1$, find $\phi\left(\frac{1}{4}\right)$, $\phi(10)$, $\phi(-10)$, $\phi(t-1)$.

2. If $H(x) = \dfrac{x^2 - 3x + 4}{x^2 + x + 1}$, find $H(0)$, $H(2)$, $H(-2)$, $H\left(\frac{1}{t}\right)$, $H(t+1)$.

3. The fact that the area, A, of a circle may be calculated from the radius, r, is expressed in the functional notation by $A = f(r)$. Give the particular form of $f(r)$ in this case.

4. If $V = F(r)$, where r is the radius and V is the volume of a sphere, give the particular form of $F(r)$.

5. If $F(s) = \dfrac{s+1}{s+2}$, find $F(3)$, $F(-1)$, $F(6)$, $F\left(\dfrac{2}{3}\right)$, $F(s^2)$.

6. If $f(x) = x^3 + 3x^2$, and $F(x) = 2x^2 + 4x - 3$, find the quotients $\dfrac{f(1)}{F(1)}$ and $\dfrac{F(2)}{f(1)}$.

7. If $\phi(x) = \dfrac{x+1}{x-1}$, find $\phi(2)$, $\phi(0)$, $\phi(\sqrt{2})$, $\phi(x+1)$.

8. If $F(x) = x^4 - 3x^2$, find $F(2)$, $F(\sqrt{2})$, $F\left(\dfrac{1}{x}\right)$, $F(\sqrt{x})$.

9. Given $y = f(x) = \dfrac{x+1}{2x-1}$. Show that $f(y)$ reduces to x.

10. If $F(x) = \dfrac{x+2}{3x+4}$, find $F(F(x))$.

15. System of coördinates. Let $X'X$ and $Y'Y$ be two straight lines meeting at right angles. Let them be considered as two number scales with the point of intersection as the zero point of each. Let P be any point in the plane. From it drop perpendiculars to the two lines. Let x represent the perpendicular to $Y'Y$, and y the perpendicular to $X'X$. If P lies to the left of $Y'Y$, x is to be considered negative. If P lies above $X'X$, then y is positive. It is clear that no matter where P is in the plane, there corresponds to it one and only one pair of perpendiculars, x and y. The lines of reference $X'X$ and $Y'Y$ are called the **coördinate axes,** and their intersection is called the **origin.** The first line is called the X-axis, and the second the Y-axis. The perpendicular to the X-axis from a given point in the plane is called the **ordinate** or y value of the point. The perpendicular to the Y-axis is called the **abscissa** or x value of the point. The coördinate axes divide the plane into four parts called **quadrants** and are conventionally numbered I, II, III, IV as in Fig. 2.

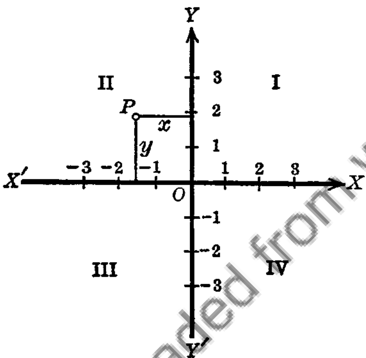

FIG. 2

If we have two numbers given we can find one and only one point P which has the first number for its abscissa and the second for its ordinate. If, for example, the numbers are 2 and -5, we measure from the origin, in the positive direction, a distance 2 on the X-axis and at this point we erect a perpendicular and measure downwards a distance 5. We have then located a point whose x is 2 and whose y is -5. This point may be represented by the symbol $(2, -5)$. The symbol (a, b) denotes a point whose abscissa is a and whose ordinate is b. The symbol $P(a, b)$ is sometimes used and is read, "the point P whose coördinates are a and b."

When a point is located in the manner described above, it is said to be **plotted.** In plotting points and obtaining the geometrical pictures we are about to make, it will be convenient to use coördinate paper, which is made by ruling off the plane into equal

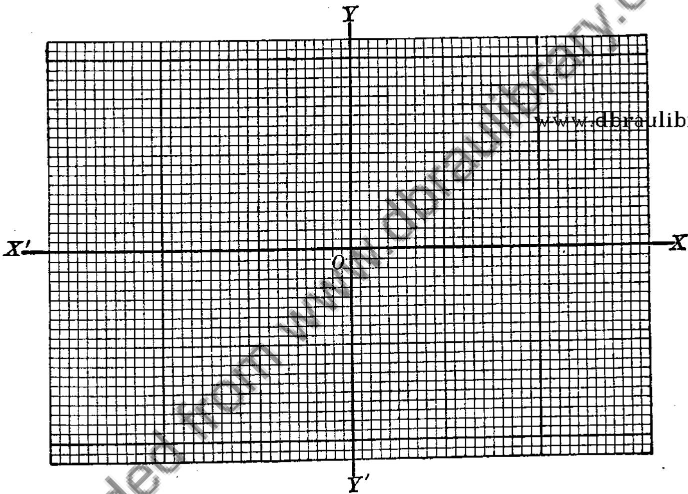

Fig. 3

squares with the sides parallel to the axes (Fig. 3). Then the side of a square may be taken as the unit of length to represent a number. To plot a point, count off from the origin along the X-axis the number of divisions required to represent the abscissa and from the point thus determined count off the number of divisions parallel to the Y-axis required to represent the ordinate. It is often convenient to take more than one side of a square for a unit of length, or to make one side represent several units.

ORAL EXERCISES

1. In what quadrant does each of the following points lie: $(1, 2)$, $(-1, -2)$, $(3, -4)$, $(-6, 7)$?

2. In which quadrant are both coördinates negative?

3. A line is horizontal and cuts the Y-axis at a distance 5 above the origin. What can you say about the ordinates of the points on the line?

4. For what points is the equation $x = 4$ true?

5. A line is bisected by the origin. One end of the line is the point $(-6, 9)$. What are the coördinates of the other end?

6. A line connects the two points $(0, 0)$ and $(4, 4)$. What are the coördinates of the mid-point of the line?

7. Describe the line each point of which has its abscissa and ordinate equal to each other but opposite in sign.

8. The abscissa of each point of a line is twice the ordinate. Describe the line.

WRITTEN EXERCISES

1. Plot the points $(2, 3)$, $(-2, 3)$, $(-2, -3)$, $(2, -3)$, $(5, 0)$, $(-7, 2)$.

2. Draw the triangle whose vertices are $(3, 5)$, $(-4, 4)$, $(1, -3)$.

3. Draw the quadrilateral whose vertices are the points, $(0, 0)$, $(1, 4)$, $(-1, 6)$, $(-4, 0)$.

4. A line joining two points is bisected at the point $(1, 0)$. If the coördinates of one end are $(8, 5)$, what are the coördinates of the other end?

5. Three corners of a rectangle are $(-1, 4)$, $(4, 4)$, $(4, -5)$. What are the coördinates of the other corner?

6. The coördinates of the vertices of a triangle are $(0, 0)$, $(6, 0)$, $(4, 4)$. What are the coördinates of the mid-points of the sides?

7. Let the X-axis represent an east and west line, the Y-axis a north and south line. The general course of the river is indicated by the following coördinates of points on the river: $(-6, -2)$, $(-5, -1.9)$, $(-4, -1.7)$, $(-3, -1.4)$, $(-2, -1.0)$, $(-1, -.5)$, $(0, .6)$, $(1, 1.3)$, $(2, 2.1)$, $(3, 3)$, $(4, 4)$, $(5, 5.1)$, $(6, 6.3)$. Map the river from $x = -6$ to $x = 6$.

16. Graph of a function. By a method analogous to that employed in Prob. 7, Art. 15, a function may be represented with reference to coördinate axes. This representation of a function is called the **graph** of the function. The graph of $f(x)$ gives a picture of the changes in $f(x)$ as x changes.

Example: Obtain the graph of $\frac{3}{2}x + 4$ for values of x between -5 and $+5$.

Let $f(x) = \frac{3}{2}x + 4$. The object is to present a picture which will exhibit the values of $f(x)$ which correspond to assigned values of x. Any assigned value of x with the corresponding value of $f(x)$ determines a point whose abscissa is x and whose ordinate is $f(x)$.

Assuming values for x and computing the corresponding values for $f(x)$, we obtain the following table.

x	0	1	2	$2\frac{1}{2}$	3	4	5	-1	$-1\frac{1}{2}$	-2	-3	-4	-5	$-\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$
$f(x)$	4	$\frac{11}{2}$	7	$\frac{31}{4}$	$\frac{17}{2}$	10	$\frac{23}{2}$	$\frac{5}{2}$	$\frac{7}{4}$	1	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	$\frac{13}{4}$	$\frac{19}{4}$	$\frac{25}{4}$

These corresponding values are plotted as coördinates of points in Fig. 4.

It should be noted that there is no limit to the number of corresponding values which we may compute and imagine plotted in a given interval along the X-axis, and further that to small changes in values of x there correspond small changes in the values of $f(x)$. These facts suggest the idea of a **continuous curve** to represent $f(x)$ much as a continuous curve is used in mapping a river. (Prob. 7, Art. 15.)

It must not, however, be assumed that all functions give continuous graphs; Art. 17, below, considers a graph made up of isolated points. The important fact for this course in algebra is that we may assume a *continuous* curve for all functions which are **polynomials** in x * and for most other functions which occur in this course, although the proof of continuity is beyond the scope of this book. That is to say, it is proved in higher analysis that a function of this type

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (n \text{ a positive integer})$$

has a continuous graph.

Hence, in finding the graph of a polynomial, when a sufficient number of points are located to suggest the general shape of a curve through them, draw a smooth curve through the points. In particular, it is proved in analytic geometry that when $n = 1$, *the graph of a function of this type is a straight line.* In the problem in hand, the graph is the straight line shown in Fig. 4.

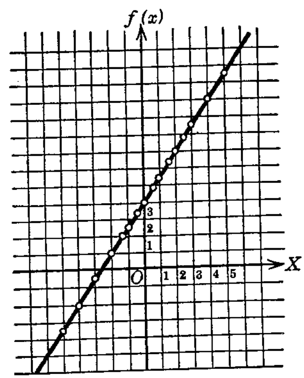

FIG. 4

17. Function defined by a table of values. Much use is made of systems of coördinates in presenting statistical results when one set of data is to be compared with another set.

The following infant mortality table is made up from the United States Life Tables of 1910. Out of 100,000 living newborn babies in each class, it shows the number of deaths during each month of the first year.

* By a polynomial in x, we mean a function of type

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

where n is a positive integer, and $a_0, a_1, \cdots, a_n$ do not contain x.

Using the numbers of the months as abscissas and the corresponding numbers in the column headed "City Males" as ordinates, we locate the upper set of points in Fig. 5. The vertical unit

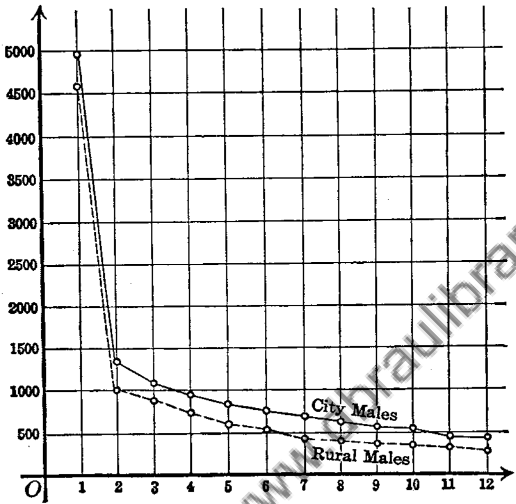

FIG. 5. — Graphical representation of infant mortality of city and rural males.

is 500. The lower set of points is given by the column headed "Rural Males." In Fig. 5 we thus present to the eye the relative infant mortality of city and country children.

MONTH OF FIRST YEAR	WHITE MALES	WHITE FEMALES	NEGRO MALES	NEGRO FEMALES	CITY MALES	RURAL MALES
1	4844	3787	7370	6380	4969	4570
2	1242	991	1977	1746	1370	997
3	1012	850	1831	1555	1091	822
4	863	740	1695	1394	941	699
5	750	648	1561	1252	835	595
6	673	578	1425	1134	755	515
7	610	526	1290	1036	694	459
8	553	486	1153	948	640	408
9	503	450	1037	874	586	363
10	457	421	937	800	537	325
11	420	390	857	725	496	296
12	399	359	802	663	466	277

The graph in this case is made up of 12 points. If we look upon the number of deaths as a function of the number of the month, this function is defined at only 12 points. The lines connecting the points in the figure are not necessary, but aid the eye to take in the whole situation. Where two sets of data are exhibited in the same diagram as in Fig. 5 the connecting lines prevent confusion of the two sets of points.

EXERCISES

Plot the graphs of the following functions.

1. $3x + 4$. **2.** $5x - 2$. **3.** $3x$. **4.** x.

5. $+\sqrt{25 - x^2}$.

Solution: We find the following table.

$x = 0$	1	2	3	4	4.5	5	Greater than 5
$\sqrt{25 - x^2} = 5$	4.9	4.6	4	3	2.18	0	Imaginary

$x = -1$	-2	-3	-4	-4.5	-5	Less than -5
$\sqrt{25 - x^2} = 4.9$	4.6	4	3	2.18	0	Imaginary

Plotting these points and drawing a smooth curve through them, we have Fig. 6.

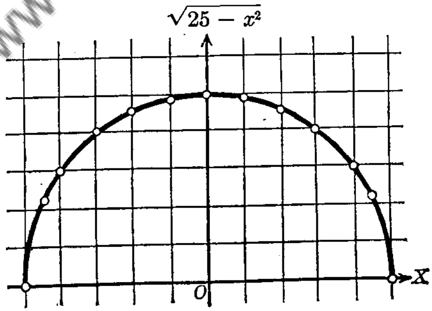

Fig. 6

6. $-\sqrt{25 - x^2}$.

7. $\sqrt{49 - x^2}$.

8. $\pm\sqrt{16 - x^2}$.

9. $x^2 - x - 20$.

10. $x^2 - 9$.

11. $20 + x - x^2$.

12. $ax^2 + bx + c$.

(*a*) when $a = 2$, $b = 1$, $c = 3$,

(*b*) when $a = -2$, $b = -1$, $c = -3$.

(Plot points for $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, -\frac{1}{2}, -1, -\frac{3}{2}, -2$.)

13. $\frac{1}{x}$. (Plot points for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, 3, 5, 10$.)

14. From the table on page 28, show graphically on the same diagram the infant mortality of "white males" and of "negro males."

15. Exhibit graphically on the same diagram the infant mortality of "white males" and "white females."

16. Exhibit graphically on the same diagram the infant mortality of "negro males" and "negro females."

17. The breaking strength of ordinary manila rope is given by the formula $B = 7100\,D^2$ where B is the breaking weight in pounds and D is the diameter of the rope. Exhibit this formula graphically for the diameters $\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}$, 1, $1\frac{1}{8}$, $1\frac{1}{4}$, $1\frac{3}{8}$, $1\frac{1}{2}$, $1\frac{5}{8}$, $1\frac{3}{4}$, $1\frac{7}{8}$, 2 inches.

18. The following table taken from a jewelry catalogue gives the price of diamonds of the same quality for various weights. From this table give a graphical representation of the price of diamonds.

Weight in carats	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
Price in dollars	30	40	50	65	80	95	110	130	150	170

Weight in carats	0.65	0.70	0.75	0.80	0.90	1.00	1.25	1.50	1.75	2.00
Price in dollars	190	210	230	250	285	325	400	500	600	700

19. The morning and evening temperatures of a pneumonia patient were as follows: 99°, 103.2°, 105°, 103.6°, 104.2°, 105°, 104°, 105°, 103°, 104.2°, 102.3°, 97.6°, 97.4°, 98.2°, 99°, 98.2°, 98.7°, 98.4°. Give a graphical representation.

Hint: To save work in handling large numbers, 90 may be subtracted from each of the above numbers and the differences plotted. Or "degrees of fever" may be plotted — that is, degrees above 98.6°.

20. The trend of gasoline consumption in the United States from 1919 through 1936 is given by the equation

$$y = 2532 + 951.2x$$

where x is the time in years measured from 1918, y is the annual consumption in millions of gallons. Represent graphically.

21. If F represents the length of a man's foot in inches and s is the size of the shoe, then

$$F = \frac{s}{3} + 8.$$

Represent graphically the relation between length of foot and size of shoe for sizes 1 to 12.

22. The postage on first-class mail matter is three cents per ounce or fraction thereof. With weights for abscissas and number of cents for ordinates exhibit this postage rate graphically.

18. Zeros of a function. By a "**zero of $f(x)$**" is meant a value of x such that the corresponding value of $f(x)$ is zero. Thus 3 and -1 are zeros of the function $x^2 - 2x - 3$, and ± 5 are zeros of $\sqrt{25 - x^2}$. Stated graphically, the "real zeros of $f(x)$" are the abscissas of the points where the graph crosses the X-axis. In Figs. 6 and 7 the graphs and the zeros of $\sqrt{25 - x^2}$ and $x^2 - 2x - 3$ are shown. One of the main problems of algebra is the development of methods for finding the zeros of functions. The graphical solution of this problem, so far as real zeros are concerned, consists in finding where the graph crosses the X-axis. One of the advantages of the graphical method of dealing with functions is that it presents to the eye the zeros of a function.

EXERCISES

Plot and find the real zeros of the following functions.

1. $x^2 - 2x - 3$.

Solution: Compute the table.

$x =$	-3	-2	-1	0	1	2	3	4	5	$\cdots$
$x^2 - 2x - 3 =$	12	5	0	-3	-4	-3	0	5	12	$\cdots$

Plotting these points and drawing a smooth curve through them, we have Fig. 7. The graph crosses the X-axis at -1, and 3, which are therefore the zeros of $x^2 - 2x - 3$.

2. $5x - 4$.

3. $2x + 2$.

4. $x^2 - 5x + 4$.

5. $4x - x^2$.

6. $6x^2 + x - 1$.

7. $5x - 4 - x^2$.

8. $x^3 + 3x^2 - x - 3$.

9. $x^3 - 9x$.

10. Between what integers does each of the real zeros of $8x^3 - 20x^2 - 2x + 5$ lie?

11. Show that $x^2 + x + 1$ has no real zeros.

12. Show that $\dfrac{1}{x+1}$ has no real zeros.

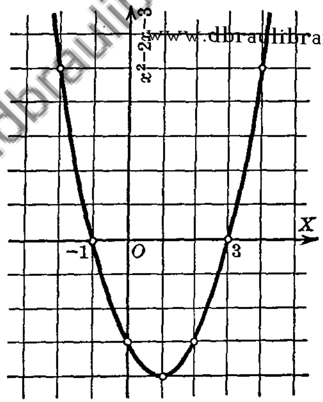

FIG. 7

CHAPTER IV

EQUATIONS AND THEIR SOLUTIONS

19. Equations and identities. A statement that one expression is equal to another expression is called an **equality.** The two expressions are called the **members** of the equality. There are two classes of equalities, — **identical equalities** or **identities,** and **conditional equalities** or **equations.** An identity is defined in Art. 7. It is there stated that the two members of an identity are equal for all values of the symbols for which the expressions are defined. Thus,

$$x^2 - a^2 = (x - a)(x + a), \qquad 5a = 10a - 5a$$

are identities. But in the equality $x - 5 = 4 - 2x$, the two expressions $x - 5$ and $4 - 2x$ are equal only when x has the value 3. An equality of this kind, in which the members can be equal only for particular values of the letters involved, that is, are not equal for all values, is sometimes called **conditional equality.** In this book we shall use the term **equation** to mean conditional equality. When it seems necessary to indicate that an equality is an identity and not a conditional equality, we shall use the sign $\equiv$ instead of the sign $=$ between the members. But the sign $=$ will be used for both identities and equations when this usage can lead to no confusion.

ORAL EXERCISES

Which of the following equalities are identities?

1. $x - 4 = 0.$

2. $(x - 1)(x - 3) = x^2 - 4x + 3.$

3. $\dfrac{x^2 - 9}{x - 3} = x + 3.$

4. $(x - 1)(x - 3) = x^2 - 5x + 6.$

5. $x^2 = (x - 4)^2 + 8x - 16.$

6. $\dfrac{1}{1 - x} = 1 + x + x^2 + \dfrac{x^3}{1 - x}.$

In equalities 3 and 6, may x take all real values?

20. Solution of an equation. In an equation there are some symbols whose values are assumed known and others whose values are unknown. These are spoken of as the **knowns** and **unknowns.** To solve an equation in one unknown is to find values of the unknown that make the two members equal. Any such value is said to satisfy the equation and is called a **solution** or **root** of the equation. A solution of an equation in more than one unknown is a set of values of the unknowns which satisfy the equation. Thus, $x = 1$, $y = 2$, is a solution of $y = x + 1$.

To solve a system of equations in any number of unknowns is to find sets of values of the unknowns which will satisfy the equations. Any such set of values is said to be a solution of the system of equations.

EXERCISES

1. Is 2 a solution of $2x + 7 = 4$?
2. Is 3 a solution of $x^2 - 5x + 6 = 0$?
3. Is 0 a root of $x^2 + x = 0$?
4. Is -1 a root of $x^3 - 3x - 5 = 0$?
5. Is $x = 1$, $y = 2$ a solution of $2x + 5y - 12 = 0$?
6. Is $x = 2$, $y = -1$ a solution of $x^2 + 2xy + 2y^2 = 1$?
7. Is $x = 1$, $y = 2$, $z = 3$ a solution of $x + 2y + 3z - 14 = 0$?

Solve the following equations for x and check by substitution:

8. $3x + 6 = 5x + 10$.
9. $7x + 9 = 2x - 6$.
10. $5x - 7 = 2x + 9$.
11. $\frac{x}{2} + \frac{x}{3} = \frac{5}{4} - x$.
12. $10 + \frac{1}{2}x = 2 + x$.
13. $\frac{5}{x - 1} = 10$.
14. $3(x - 1) = 2(x + 1)$.
15. $3(x + 1) + x^2 = x^2 + 12$.
16. $(x + 1)(x + 3) = x(x - 2)$.
17. $bx + b = 6b$.
18. $3(a - x) = 10a$.
19. $5x - \frac{8}{11} + \frac{2(10 - x)}{2} = x$.

21. Equivalent equations. Two equations or two systems of equations are said to be **equivalent** when they have the same solutions; that is, when each equation or each system is satisfied by the solutions of the other. Thus, the equations $x - 2 = 0$ and $3x - 6 = 0$ are equivalent, the second being derived from the first by multiplying both members by 3. Again, the equations $x^2 - 5x + 6 = 0$, and $-10x^2 = -50x + 60$, are equivalent. The second can be obtained from the first by performing the following operations on both members.

(1) Multiply both members by -10.
(2) Add $-50x + 60$ to both members.

It must not, however, be inferred when the same operation is performed on the two members of an equation, that there necessarily results an equivalent equation. The following examples will show that this is an unwarranted inference.

EXAMPLES

1. Consider the equation $3x = x + 4$. (a)
Square both members, $9x^2 = x^2 + 8x + 16$. (b)
The equation (b) is satisfied by 2 and -1, while (a) is satisfied by 2 and not by -1. Hence, (a) and (b) are not equivalent.

2. Consider the equation $3x + 2 = 5x - 8$. (c)
Multiply both members by $(x - 1)$,

$$(x - 1)(3x + 2) = (x - 1)(5x - 8). \quad (d)$$

Equation (d) is satisfied by 1 and 5, while (c) is satisfied only by 5. Hence, (c) and (d) are not equivalent.

3. Consider the equation $\sqrt{1 - x} - x = -1$. (e)
First, add x to each member, then square both members. There results

$$1 - x = 1 - 2x + x^2. \quad (f)$$

Equation (f) is satisfied by $x = 0$ and $x = 1$; but $x = 0$ does not satisfy (e). Hence, (e) and (f) are not equivalent.

4. Consider the system of equations

$$\left.\begin{aligned} x + y &= 15, \\ x - y &= 5. \end{aligned}\right\} \quad (g)$$

Multiply the members of the first equation of (g) by x, the second by y. There results

$$\left.\begin{aligned} x(x + y) &= 15x, \\ y(x - y) &= 5y. \end{aligned}\right\} \quad (h)$$

This system (h) is satisfied by the four pairs of numbers (10, 5),* (0, 0), (0, $-$ 5), (15, 0), but (10, 5) is the only one of these pairs which will satisfy (g).

These simple examples show that the same operation performed on the two members of an equation does not necessarily lead to an equation equivalent to the original one.

It is manifestly important to know whether an equation is equivalent to that from which it is derived; and if non-equivalent, whether it contains at least all the solutions of the original equation.

The following operations which the student has often performed in elementary algebra lead to equivalent equations:

(a) *Adding the same number to or subtracting the same number from both members.*

* The notation (10. 5) means $x = 10$, $y = 5$. (See Art. 15.)

(b) *Multiplying or dividing both members by the same known number provided this number is not equal to zero.*

(c) *Changing the signs of all the terms.*

If a derived equation contains all the roots of the original equation and some others, we shall call it **redundant.** If the derived equation lacks some roots of the original equations, we shall call it **defective.** The student should always be on his guard against treating two equations as necessarily equivalent simply because the one has been derived from the other.

22. Operations that lead to redundant equations. The following operations on the two members of an equation lead, in general, to redundant equations:

(a) Multiplying both members of the given equation by a function of the unknown that has a zero (Art. 18).

Example 1. Consider the equation $7x + 3 = 9x - 4$. (a)

The root or solution is $x = \frac{7}{2}$.

Multiply each member by $(2x - 3)$. We have

$$(2x - 3)(7x + 3) = (2x - 3)(9x - 4). \qquad (b)$$

Equation (b) has roots $\frac{7}{2}$ and $\frac{3}{2}$, but $\frac{3}{2}$ is not a root of equation (a).

Example 2. Consider the equation $x - 1 = 0$. (c)

The solution is $x = 1$.

Multiplying each member by x, we have

$$x^2 - x = 0. \qquad (d)$$

Equation (d) has roots 0 and 1, but 0 is not a root of (c).

(b) Raising both members to the same integral power.

Example 1. Take the equation $3x = x + 4$. (e)

Squaring each member, we have

$$9x^2 = x^2 + 8x + 16. \qquad (f)$$

Equation (f) has roots 2 and -1, but -1 is not a root of equation (e).

Example 2. Take the equation $-\sqrt{x} = 1$. (g)

There is no value of x that satisfies (g).

Squaring both members, we have

$$x = 1. \qquad (h)$$

While 1 thus satisfies equation (h), it does not satisfy (g).

NOTE: It is common practice to consider $\sqrt{x}$ as representing the positive square root of x. When both roots are meant we write $\pm\sqrt{x}$. Thus $\sqrt{4} = 2$, $\pm\sqrt{4} = \pm 2$.

is of degree two in x, one in y, one in z, one in y and z, two in x and z, three in x and y, three in x, y, and z.

25. Rational integral equations. An equation, in one unknown x, of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \ (a_0 \neq 0),$$

in which the left-hand side is a rational integral expression of degree n in x (a polynominal of degree n in x) is called a **rational integral equation of degree n** in x. Thus,

$$2x^2 - 5x = 0,$$

and

$$\tfrac{3}{2}x^2 + 7x - \tfrac{2}{3} = 0$$

are rational integral equations of degree 2 in x.

As an extension of this definition, a rational integral equation in unknowns $x, y, z, \ldots$ is a statement that two rational integral expressions involving x, y, z, . . . are equal. For example,

$$3x^2 - 5xy - y^3 = 2xz^2 - 4xyz \qquad (1)$$

is a rational integral equation in x, y, and z.

The degree of a rational integral equation in certain unknowns is defined as the degree of a term whose degree is equal to or greater than that of any other term in the equation. Thus, equation (1) is of degree two in x, three in y, three in x and z, three in x, y, and z.

In this course, the term degree is applied to equations only when they are in the rational integral form.

We sometimes speak of the degree of an equation without mentioning to what letters we refer. In this case, it is to be understood that we mean the degree in all the unknowns.

Equations of the first, second, third, fourth, and fifth degrees are called **linear, quadratic, cubic, quartic,** and **quintic** equations respectively.

ORAL EXERCISES

Give the degree of each of the following equations.

1. $ax + by + c = 0.$

2. $ax^2 + bxy + cy^2 = 0.$

3. $5x^2 + x^2y^2 - y = 0.$

4. $y^4 + 2x^2y^3 - 3x^3y - x^3 = 0.$

5. Give the degree of the expression $ax^5 - 4mx^2y^2 - 3nxy + y^3$ in x. In y. In x and y.

6. Give the degree of the equation $10x^5 - 4ax^2yz - 3xyz + by^2 = 5x^4 - 2x^2y^2$ in x. In y. In z. In y and z. In x and z. In x and y. In x, y, and z.

7. Given an equation whose members are rational integral functions of x. If you multiply the members by $x - a$ where a is not a root of the given equation, what root is introduced into the derived equation? Illustrate with the given equation $x = b$.

8. If $x - a$ is a factor of each member of a given equation, what root of the given equation is, in general, lacking in the equation obtained by dividing the members of the given equation by $x - a$? Illustrate with the given equation $x(x - a) = b(x - a)$.

9. Give examples of rational integral equations in one unknown, x, which are (1) linear, (2) quadratic, (3) cubic, (4) quartic, (5) quintic.

CHAPTER V

SYSTEMS OF LINEAR EQUATIONS

26. Type form. An equation of the form

$$ax + by + c = 0, \tag{1}$$

is called a linear equation in two unknowns.

When $b \neq 0$, it can be put into the form $y = -\frac{ax}{b} - \frac{c}{b}$. (2)

Since in (2) we may assign to x any value and compute a corresponding value for y, the equation defines y as a function of x in accordance with our definition of a mathematical function (Art. 13).

The graph of the linear function is a straight line (Art. 16). The straight line representing the function $-\frac{ax}{b} - \frac{c}{b}$ is also the locus of all points whose coördinates satisfy the equation $y = -\frac{ax}{b} - \frac{c}{b}$. Hence, the graphical representation of the equation $ax + by + c = 0$ is a straight line.

EXERCISES

Graph each of the following equations.

1. $x - y = 1$.

Solution: This equation may be written in the form

$$y = x - 1.$$

The graph of the linear function $x - 1$ is the line shown in Fig. 8, and is by definition the graph of the equation $x - y = 1$. Since we know the graph to be a straight line, it is necessary to plot two points only, and to draw a straight line through them. The farther apart the two points are, the more accurate the graph is likely to be.

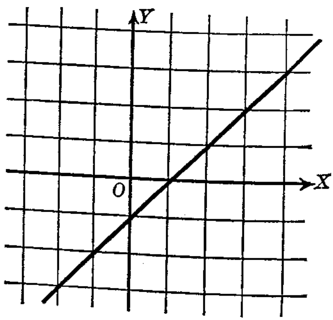

FIG. 8

2. $x + y = 1$.

3. $x - 2y = 1$.

4. $2x - y = 1$.

5. $3x + 4y = 5$.

6. $4x + 3y = 0$.

7. $4x - 3y = 0$.

8. $\frac{x + y}{2} + \frac{x - y}{3} = 1$.

27. Graphical solution of a set of linear equations. As stated in Art. 26, the graph of any linear equation in x and y is a straight line. Any such equation is satisfied by an indefinitely large number of pairs of values of x and y, that is, by the coördinates of all points on its graph. In the graphical solution of the system of two equations, we seek the coördinates of points common to the graphs of the two equations.

As the graphs are two straight lines, three cases arise:

(1) In general, two lines intersect in one and only one point.

(2) Two lines may be parallel, and thus have no point in common.

(3) Two lines may be coincident, and thus have an indefinitely large number of points in common.

Corresponding to these three cases, a set of two linear equations has, in general, one and only one solution, but it may have no solution or an indefinitely large number of solutions. When the graphs are two parallel lines, there is no pair of numbers which satisfies both equations, and the equations are said to be **incompatible** or **inconsistent.** When the graphs are two coincident straight lines, the two equations of the system are equivalent (Art. 21). Examples of each case are given in the following exercises.

EXERCISES

Find the solutions of the following equations by plotting the graphs.

1. $x - y + 1 = 0,$
$4x + y - 16 = 0.$
See Fig. 9.

2. $x + y - 4 = 0,$
$2x - y + 1 = 0.$

3. $x + y - 4 = 0,$
$x + y - 2 = 0.$

4. $2x - 3y = 0,$
$2x - 3y = 1.$

5. $x - 3y = 4,$
$2x - 6y = 8.$

6. $x - y = 0,$
$5x - 5y = 0.$

7. $3x - 2y = 12,$
$5x + 3y = 1.$

8. $4x - 12y - 44 = 0,$
$8x + 11y - 18 = 0.$

9. $5x + 3y + 9 = 0,$
$3x - 4y + 17 = 0.$

10. $\frac{x}{2} + \frac{y}{3} = 4,$
$\frac{x}{2} - \frac{y}{3} = 0.$

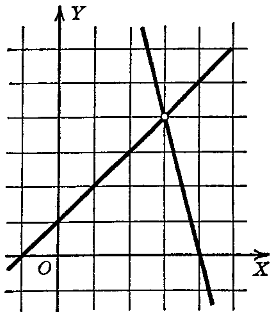

FIG. 9

11. Graph $2x + 3y = 1$. Multiply both sides of this equation by 3 and graph the resulting equation. Compare the graphs.

12. Graph $2x + 3y = c$ for two different given values of c. Compare the graphs.

28. Solution by elimination. The process of deriving from a system of n equations, a new system of $n - 1$ equations with one fewer unknowns is called **elimination.** Thus, when the given system consists of two equations in two unknowns, the new system consists of a single equation in one unknown.

Elimination by **addition or subtraction** is illustrated in the following example.

Example. Solve

$$2x + 3y = 4,$$
$$3x - 4y = 5.$$

Solution. To eliminate x we first multiply the members of the first equation by 3 and those of the second equation by 2, obtaining

$$6x + 9y = 12,$$
$$6x - 8y = 10.$$

Subtracting,

$$17y = 2. \quad \text{Then } y = \frac{2}{17}.$$

Substituting $\frac{2}{17}$ for y in either equation, say in the first, we get

$$2x + 3 \cdot \frac{2}{17} = 4, \quad \text{or} \quad x = \frac{31}{17}.$$

Check: Substituting $\frac{31}{17}$ for x, and $\frac{2}{17}$ for y in the two equations gives

$$2 \cdot \frac{31}{17} + 3 \cdot \frac{2}{17} = \frac{62}{17} + \frac{6}{17} = \frac{68}{17} = 4$$
$$3 \cdot \frac{31}{17} - 4 \cdot \frac{2}{17} = \frac{93}{17} - \frac{8}{17} = \frac{85}{17} = 5.$$

We can solve the equations also by eliminating y, which is done as follows: Multiply the members of the first equation by 4, and of the second by 3, obtaining

$$8x + 12y = 16,$$
$$9x - 12y = 15.$$

Adding

$$17x = 31, \quad \text{or} \quad x = \frac{31}{17},$$

and substitution in the first equation gives $y = \frac{2}{17}$.

Elimination by **substitution** consists of expressing one unknown in terms of the other from one equation, and substituting this result in the other equation, thus obtaining one equation in one unknown.

Example. Solve by substitution

$$2x + 3y = 4$$
$$3x - 4y = 5.$$

Solution: From the second equation we obtain x in terms of y,

$$3x = 5 + 4y,$$

or

$$x = \frac{5 + 4y}{3}.$$

Substituting $\frac{5+4y}{3}$ for x in the first equation gives

$$2 \cdot \frac{5+4y}{3} + 3y = 4,$$

or

$$10 + 8y + 9y = 12,$$

whence

$$17y = 2, \quad \text{or} \quad y = \frac{2}{17}.$$

and

$$x = \frac{5+4y}{3} = \frac{5 + 4 \cdot \frac{2}{17}}{3} = \frac{93}{3 \cdot 17} = \frac{31}{17},$$

which checks with the solution by the method of elimination by addition or subtraction.

EXERCISES

Solve by elimination by addition or subtraction.

1. $2x - y = 5,$
   $x - 2y = -2.$

2. $5x + 6y = 22,$
   $x - 4y = -6.$

3. $5x + 2y - 4 = 0,$
   $10x - 4y + 48 = 0.$

4. $3x - 2y - 3 = 0,$
   $2x - y - 4 = 0.$

5. $0.7x + 0.3y = 0.68,$
   $0.3x + 0.7y = 0.92.$

6. $ax + by = c,$
   $dx + ey = f.$

7. $\frac{x+a}{a} + y = 1,$
   $x + \frac{y+b}{b} = 1.$

8. $mx + ny = 2mn,$
   $nx + my = m^2 + n^2.$

9. Solve exercise **1** by substitution.

10. Solve exercise **2** by substitution.

29. Solution by determinants.

Let

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

be two linear equations in two unknowns. Multiply the members of the first by b_2, and those of the second by $-b_1$. Adding the members of the two resulting equations, we obtain

$$(a_1b_2 - a_2b_1)x = (b_2c_1 - b_1c_2),$$

or

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

In a similar manner, by multiplying the first and second equations by $-a_2$ and a_1 respectively, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

We note that the denominators of the above fractions are alike. The denominator may be denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is called a **determinant.** Since it has two rows and two columns, it is said to be of the second order. The letters a_1, b_1, a_2, b_2, are called the **elements** of the determinant, and a_1, b_2, constitute the **principal diagonal.** A determinant of the second order then represents the number which is obtained by subtracting from the product of the terms in the principal diagonal, the product of the other two terms. Thus,

$$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz, \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

Using the determinant notation, we may now write the solutions of our equations in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

We note that the numerator of the solution for x is obtained from the denominator by substituting in place of a_1, a_2, which are the coefficients of x in the equations to be solved, the known terms c_1, c_2. In a similar manner, in the numerator of the solution for y we replace b_1, b_2 by c_1, c_2 respectively.

EXERCISES

Solve the following equations by the use of determinants.

1.
$$2x + 3y = 4,$$
$$3x - 4y = 5.$$

Solution:

$$x = \frac{\begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -4 \end{vmatrix}} = \frac{-16 - 15}{-8 - 9} = \frac{-31}{-17} = \frac{31}{17}.$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -4 \end{vmatrix}} = \frac{10 - 12}{-8 - 9} = \frac{-2}{-17} = \frac{2}{17}.$$

2. $x + y = 3,$
$2x + 3y = 1.$

3. $3x + 5y = 12,$
$x + 2y = 5.$

4. $3x - 2y = 7,$
$2x + y = 21.$

5. $2x - 4y = 1,$
$70x + 30y = 1.$

6. $x + \frac{y}{2} = 4,$
$\frac{x}{3} + \frac{y}{4} = \frac{7}{6}.$

7. $\frac{(x + y)}{2} + (x + y) = 1,$
$(x + y) - \frac{(x - y)}{2} = -\frac{1}{2}.$

8. $\frac{3}{x} + \frac{5}{y} = 12,$
$\frac{1}{x} + \frac{2}{y} = 5.$

9. $\frac{1}{x} + \frac{1}{y} = 3,$
$\frac{2}{x} + \frac{3}{y} = 1.$

Hint: Solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

10. In solving a system of equations

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$$

by determinants, show that if the determinant in the denominator and the determinants in the numerators are all zero, then the two equations are equivalent (Art. 21).

30. Systems of three linear equations in three unknowns. A system of three linear equations in three unknowns may be solved by the method of elimination, but the use of determinants (Art. 32) gives a more systematic solution. There are special systems in which there are no solutions, or infinitely many solutions. Such systems are not considered in this chapter. The following example illustrates the solution by elimination.

Example.

$$\begin{aligned} x - 2y - z &= -7, \\ 2x + y + z &= 0, \\ 3x - 5y + 8z &= 13. \end{aligned}$$

Solution: Add together the first two equations to eliminate z:

$$\begin{array}{r} x - 2y - z = -7, \\ 2x + \ \ y + z = 0 \\ \hline 3x - \ \ y \qquad = -7. \end{array}$$

Multiply the second of the given equations by 8 and from the result subtract the third equation:

$$\begin{array}{r} 16x + \ \ 8y + 8z = 0, \\ 3x - \ \ 5y + 8z = 13 \\ \hline 13x + 13y \qquad = -13, \end{array}$$

or

$$x + y = -1.$$

We have now reduced the system to that of two equations in x and y, i.e.

$$3x - y = -7,$$
$$x + y = -1.$$

Eliminate y by addition, obtaining $4x = -8$, or $x = -2$. Substituting $x = -2$ in $x + y = -1$, we find $y = 1$. Substituting $x = -2$, $y = 1$ in $x - 2y - z = -7$, we get $z = 3$.

Check.

$$(-2) - 2(1) - (3) = -7,$$
$$2(-2) + (1) + (3) = 0,$$
$$3(-2) - 5(1) + 8(3) = 13.$$

EXERCISES

Solve the following systems of equations:

1. $3x + 2y - z = 4,$
$5x - 3y + 2z = 5,$
$6x - 4y + 3z = 7.$

2. $x + y + z = 1,$
$3x + 2y + 7z = 1,$
$15x - 4y + 8z = 18.$

3. $\dfrac{14}{x} + \dfrac{14}{y} = 23,$
$\dfrac{2}{y} + \dfrac{2}{z} = 5,$
$\dfrac{7}{x} + \dfrac{7}{z} = 8.$

Hint: Solve first for the variables $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$.

4. $x + 2y + z = 0,$
$2x + y + 2z = 3,$
$4x - 6y + 3z = 14.$

5. $x + y = 1,$
$y + z = 2,$
$z + x = 4.$

6. $ax + by = a,$
$by + cz = b,$
$ax + cz = c.$

7. $x + y = 3a,$
$x + z = 4a,$
$y + z = 5a.$

8. $\dfrac{1}{2x} + \dfrac{1}{3y} = 9,$
$\dfrac{1}{3x} + \dfrac{1}{2z} = 8,$
$\dfrac{1}{2y} + \dfrac{1}{3z} = 13.$

9. $2x + 3y + 5z = 2,$
$5x - y + 4z = 5,$
$7x - 2y + 6z = 5.$

10. $A - 5B - 6C = 2,$
$-3A - 4B + 10C = 7,$
$10A + 2B - 36C = -1.$

31. Determinants of the third order. The square array of nine numbers with bars on the sides

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is a convenient abbreviation for the expression

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3, \qquad (1)$$

and is called a determinant of the third order. As in the case of the determinant of the second order, the letters a_1, b_1, etc., are called the elements, and the letters a_1, b_2, c_3 form the principal diagonal. The expression (1) is called the expansion or development of the determinant. It is seen that each term of the expansion consists of the product of three elements, no two of which lie in the same row or in the same column. Any determinant of the third order may be easily expanded. Rewrite the first and second columns to the right of the determinant. The diagonals running

$$\left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right|$$

down from left to right give the positive terms. The diagonals running down from right to left give the negative terms.

EXERCISES

Obtain the expansions of the following determinants.

1. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1\cdot7\cdot5 + 3\cdot3\cdot1 + 4\cdot3\cdot2 - 4\cdot7\cdot1 - 3\cdot3\cdot1 - 5\cdot2\cdot3 = 1.$

2. $\begin{vmatrix} 1 & 1 & 6 \\ 2 & 0 & 3 \\ 1 & 2 & 13 \end{vmatrix}.$

4. $\begin{vmatrix} 6 & 1 & 1 \\ 3 & 3 & -1 \\ 13 & 2 & 3 \end{vmatrix}.$

6. $\begin{vmatrix} 2 & x & x \\ 1 & 1 & 1 \\ 4 & x & x \end{vmatrix}.$

3. $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}.$

5. $\begin{vmatrix} 0 & 7 & 0 \\ -2 & 3 & -10 \\ -9 & 5 & -21 \end{vmatrix}.$

7. $\begin{vmatrix} a & 0 & b \\ 0 & y & x \\ u & v & 0 \end{vmatrix}.$

32. Solution of three equations with three unknowns by determinants. Let the three equations be

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$
$$a_2x + b_2y + c_2z = d_2, \tag{2}$$
$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

Multiplying (1) and (2) by b_2 and $-b_1$ respectively and adding, we get

$$(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = d_1b_2 - d_2b_1. \tag{4}$$

Eliminating y in a similar manner from (1) and (3), we find

$$(a_3b_1 - a_1b_3)x + (c_3b_1 - b_3c_1)z = d_3b_1 - d_1b_3. \tag{5}$$

We now have two equations in two unknowns, x and z. Eliminating z from these two, we find

$$[(a_1b_2 - a_2b_1)(c_3b_1 - b_3c_1) - (a_3b_1 - a_1b_3)(b_2c_1 - b_1c_2)]x$$
$$= (d_1b_2 - d_2b_1)(c_3b_1 - b_3c_1) - (d_3b_1 - d_1b_3)(b_2c_1 - b_1c_2),$$

which after some simplification gives us

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_3b_2c_1 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3}.$$

The denominator is the development of the determinant in Art. 31, while the numerator is the same as the denominator with a_1, a_2, a_3 replaced by d_1, d_2, d_3 respectively. Hence we can write the solution for x in the form

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

provided the determinant in the denominator is not zero.

In a similar way, we can find the values of y and z.

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The denominators in the expressions for x, y, and z are the same, while the numerators are obtained from the denominators by replacing the coefficients of the unknown in question by the known terms. For example, in the numerator of y, the knowns d_1, d_2, d_3 replace b_1, b_2, b_3 respectively.

EXERCISES

Solve:

1.
$$\begin{aligned} x - 2y - z &= -7, \\ 2x + y + z &= 0, \\ 3x - 5y + 8z &= 13. \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} -7 & -2 & -1 \\ 0 & 1 & 1 \\ 13 & -5 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{-104}{52} = -2.$$

$$y = \frac{\begin{vmatrix} 1 & -7 & -1 \\ 2 & 0 & 1 \\ 3 & 13 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{52}{52} = 1.$$

$$z = \frac{\begin{vmatrix} 1 & -2 & -7 \\ 2 & 1 & 0 \\ 3 & -5 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & 5 & 8 \end{vmatrix}} = \frac{156}{52} = 3.$$

2. $$\begin{aligned} x + y + z &= 6, \\ 2x + y - z &= 3, \\ x + 2y + 3z &= 13. \end{aligned}$$

3. $$\begin{aligned} x + y + 2z &= 7, \\ 2x + 2y + 2z &= 10, \\ 3x - 2y + z &= 1. \end{aligned}$$

4. $$\begin{aligned} 3x + y - z &= 15, \\ x + 3y - z &= 17, \\ x + y - 3z &= -7. \end{aligned}$$

5. $$\begin{aligned} \frac{2}{x} + \frac{3}{y} + \frac{4}{z} &= -2, \\ -\frac{2}{x} + \frac{9}{y} - \frac{8}{z} &= 10, \\ \frac{6}{x} + \frac{6}{y} + \frac{12}{z} &= -7. \end{aligned}$$

6. $$\begin{aligned} \frac{1}{x} + \frac{2}{y} &= 0, \\ \frac{3}{x} - \frac{1}{z} &= 2, \\ \frac{4}{y} + \frac{3}{z} &= 1. \end{aligned}$$

7. $$\begin{aligned} 2x + 3y - 3z &= 36, \\ x + 2y + 3z &= 13, \\ 3x + 4y &= 1. \end{aligned}$$

8. $$\begin{aligned} x - y + z &= -9, \\ 3y - x - z &= 51, \\ 7z - y + 2x &= 63. \end{aligned}$$

9. $$\begin{aligned} x + y - z &= 0, \\ x - y &= 2b, \\ x + z - 3a - b &= 0. \end{aligned}$$

10. $$\begin{aligned} x + \frac{y}{2} + \frac{z}{3} &= 9, \\ \frac{1}{2}(x + z) + y &= 7, \\ \frac{1}{4}(x - z) - 2y + 3 &= 0. \end{aligned}$$

11. $$\begin{aligned} \frac{x + y}{2} + \frac{z}{3} &= \frac{7}{3}, \\ \frac{x + z}{2} + \frac{y}{3} &= \frac{8}{3}, \\ \frac{y + z}{2} + \frac{x}{3} &= 3. \end{aligned}$$

MISCELLANEOUS PROBLEMS

1. Two numbers are written with the same two digits. The difference between the two numbers is 36 and the sum of the digits is 12. What are the numbers?

Hint: Let x = one digit, y the other. Then the two numbers are $10x + y$ and $10y + x$.

2. If a two-digit number is divided by the digit in units' place, the quotient is 15. Subtracting 18 from the number reverses the digits. What is the number?

3. A was m times as old as B a years ago, and will be n times as old as B in b years from now. Find the ages of each, first when $a = 18$, $m = 7$, $b = 12$, and $n = 2$, and then in terms of a, b, m, and n.

4. A linear function of x takes on the value 5 when $x = 4$, and 23 when $x = 10$. What is the function?

5. A quadratic function of x in the form $ax^2 + bx + c$ takes on the value 12 when $x = -1$, -2 when $x = 1$, and 0 when $x = 2$. What is the function?

6. If a man's diet should consist of 4.6 ounces protein, 2.1 ounces fat, and 18.1 ounces carbohydrate, and be made of cereal, milk, and eggs, how many ounces of each are needed? According to the United States Department of Agriculture cereal contains 14% protein, 2% fat, 72% carbohydrate; milk 3% protein, 4% fat, 5% carbohydrate; eggs 15% protein, 10% fat.

Problems Pertaining to Mensuration

7. The sum of the three angles of a triangle is 180°. The largest angle is three times as large as the smallest one and equal to the sum of the two smaller angles. Find the three angles.

8. Two angles are supplementary, and one exceeds the other by 64°. Find the angles.

9. The perimeter of a rectangular field is 160 rods. If the length exceeds the width by 20 rods, find the length and width.

10. If the sides of a rectangular field were each increased 10 rods, the area would be increased 900 square rods. If the length were decreased 10 rods and the width increased 10 rods, the area would be increased 100 square rods. Find the area of the field.

Problems Pertaining to Finance

11. A man has $35,000 at interest. For one part he receives $3\frac{1}{2}\%$ interest and for the other 4%. His income from this investment is $1300 per year. How is the capital divided?

12. How would the $35,000 of problem 11 be divided if the interest rates were 3% and 4%?

13. What is the capital of a person whose income is $2680 when he has $\frac{1}{3}$ of it invested at 4%, $\frac{2}{5}$ of it at $4\frac{1}{2}\%$, and the rest at 5%?

14. A man left to his three sons his property worth \$18,600. The oldest son had already received \$1500 toward his education, the second son \$700, and the third son \$300. The father wanted this inequality corrected by the amounts named in the final settlement. How much did each son get in the final settlement?

15. A man distributes P dollars among n persons of three classes, the first class receiving a dollars each, the second b dollars each, and the third c dollars each. The first class receives the same total amount as the other two classes. Find in terms of P, n, a, b, and c the number that received a, b, and c dollars.

Problems Pertaining to Averages

16. Four numbers have the property, that when successively the arithmetic average of three of them is added to the fourth, the numbers 21, 23, 27, 31 result. What are the numbers?

17. To find the average grade of a freshman in mathematics, his grade in analytic geometry is multiplied by 5, his grade in algebra by 3, and his grade in trigonometry by 2, and the sum of the three products is divided by 10. This gives 87 for the average grade. If the grades in analytic geometry and algebra had been interchanged, his average grade would have been 89. If the three studies had all counted the same number of credits, his grade would have been 88. What is the grade in each of the three studies?

18. A student has three grades which give an average 78. The sum of the two extreme grades exceeds the intermediate grade by 80. The highest grade exceeds the intermediate by 18. What are the grades?

19. Two boys have the same average grade of 87 in the same three subjects. For the first boy the difference between the highest and lowest grades is 9. For the second boy this difference is 20. The average of the highest and lowest grades is 87.5 for the first boy and 87 for the second boy. What are the grades?

Problems Pertaining to Mixtures

20. One bar of metal is 20% pure silver and another is 12% pure silver. How many ounces of each bar must be used, if, when the parts are melted together, a bar weighing 40 ounces is obtained, of which 15% is pure silver?

21. What amounts of silver 72% pure and 84.8% pure must be mixed to get 8 ounces of silver 80% pure?

22. The crown of Hiero of Syracuse was part gold and part silver. It weighed 20 pounds, and lost 1.25 pounds when weighed in water. How much gold and how much silver did it contain if 19.25 pounds of gold and 10.5 pounds of silver each lose a pound when weighed in water?

Explanation of weight in water. A body like a piece of gold or silver when weighed in water loses an amount of weight equal to the weight of water displaced. Thus, if x and y denote the weights of gold and silver respectively,

$$\frac{x}{19.25} + \frac{y}{10.5} = 1.25.$$

23. A composition of tin and copper containing n cubic inches weighs p ounces. The weight of a cubic inch of tin is t ounces, and that of a cubic inch of copper is c ounces. What is the number of cubic inches of tin?

Problems Pertaining to Uniform Motion

24. Two automobiles left a garage at the same time going in opposite directions. The first traveled 8 miles more per hour than the second. At the end of 5 hours they were 280 miles apart. What were their speeds per hour?

25. Two runners are practicing on a circular track 126 yards in circumference. When running in opposite directions they meet every 13 seconds. Running in the same direction, the faster passes the slower every 126 seconds. How many minutes does it take each to run a mile?

Problems Pertaining to Physics

26. In Wilson and Gray's determination of the temperature of the sun the Fahrenheit reading of the temperature is 5552 more than the centigrade reading. What is the centigrade reading?

27. If h represent the height in meters above sea level, and b represent the reading of a barometer in millimeters, it is known that $b = k + hm$, where k and m are constants. At a height 120 meters above sea level the barometer reads 751; at height 769 meters it reads 695. What is the formula showing the relation between b and h?

28. The relation between the boiling point w of water in degrees Fahrenheit and h, the height in feet above sea level, is known to be of the form $h = x - wy$, where x and y are numbers to be determined by experiment. It is observed at the height 2200 feet that the boiling point is 208° F. At sea level the boiling point is 212° F. What is the formula showing the relation between w and h?

29. It is required to find the amount of expansion of a brass rod for a rise in temperature of one degree centigrade, also the length of the rod at a temperature 0°. If c represent the expansion, and b_0 the length required, it is known that $b = ct + b_0$, where b is the length of the bar at the temperature t. When $t = 20°$, the length of the rod is 1000.22; when $t = 60°$, the length is 1001.65.

Problems Involving Determinants

30. Develop

$$\begin{vmatrix} a_1 & a_1 & a_2 \\ b_1 & b_1 & b_2 \\ c_1 & c_1 & c_2 \end{vmatrix}$$

31. Show that $\begin{vmatrix} a_1 & b_1 + b_2 \\ a_2 & c_1 + c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & c_1 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 \\ a_2 & c_2 \end{vmatrix}$

32. Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

33. Solve for x:
$$\begin{vmatrix} x & 1 \\ 14 & 7 \end{vmatrix} = 0.$$

34. Solve for x:
$$\begin{vmatrix} x & x & 1 \\ 2 & 4 & 5 \\ 3 & 4 & 6 \end{vmatrix} = 0.$$

35. Solve for x and y the system of equations:
$$\begin{vmatrix} x & 1 & y \\ 2 & -3 & 1 \\ -1 & 2 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} x & 0 & 1 \\ y & -1 & 0 \\ -3 & 1 & 1 \end{vmatrix} = 0.$$

CHAPTER VI

EXPONENTS AND RADICALS

33. Introduction. In certain exercises in Chapter II, we have performed multiplications by means of the law of exponents, $a^m a^n = a^{m+n}$, and divisions by means of the law, $a^m \div a^n = a^{m-n}$, for special values of m and n.

We shall soon find it convenient to make use of more general exponents and their laws of operation.

34. Positive integral exponents. Definition. — The expression a^x is read "*a exponent x*" or the "xth power of a." When x is a positive integer, a^x is a short way of writing $a \cdot a \cdot a \cdots$ to x factors.

Laws of exponents:

I. $$a^m a^n = a^{m+n}.$$

For, if m and n are positive integers, by the associative law of multiplication,

$$\begin{aligned} a^m a^n &= (a \cdot a \cdot a \cdots m \text{ times})(a \cdot a \cdot a \cdots n \text{ times}) \\ &= a \cdot a \cdot a \cdots m + n \text{ times} \\ &= a^{m+n}. \end{aligned}$$

Illustrations: $$5^3 \cdot 5^4 = 5^7, \quad 3 \cdot 3^2 \cdot 3^3 \cdot 3^5 = 3^{11}.$$

II. $$(a^m)^n = a^{mn}.$$

Illustration: $$(3^4)^5 = 3^{20}.$$

III. $$(abc \cdots)^m = a^m b^m c^m \cdots.$$

Illustration: $$(3 \cdot 5 \cdot 7 \cdot 4)^2 = 3^2 \cdot 5^2 \cdot 7^2 \cdot 4^2.$$

IV. $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Illustration: $$\left(\frac{3}{7}\right)^3 = \frac{3^3}{7^3}.$$

V. $$\frac{a^m}{a^n} = a^{m-n}, \ (m > n).$$

Illustration: $$\frac{5^6}{5^4} = 5^2.$$

VI. $$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \ (m < n).$$

Illustration: $$\frac{5^4}{5^6} = \frac{1}{5^2}.$$

EXERCISES

What number should be written in each of the parentheses in the following?

1. $2^4 \cdot 2^3 = 2^{(\ )}$.
2. $4 \cdot 4^2 \cdot 4^5 = 4^{(\ )}$.
3. $(3^2)^5 = 3^{(\ )}$.
4. $\left(\frac{5^4}{5^2}\right)^3 = 5^{(\ )}$.
5. $\left(\frac{3}{2}\right)^6 \cdot \left(\frac{2^2}{3}\right)^3 \cdot 2^4 = 2^{(\ )} \cdot 3^{(\ )}$.

Perform the indicated operations and simplify the results when possible. Fractions should be reduced to lowest terms.

6. $5^2 \cdot 5^4$.
7. $a^5b^2 \cdot a^7b^3$.
8. $\frac{a^8}{a^4}$.
9. $\frac{a^9}{a^{12}}$.
10. $h^n \cdot h^3$.
11. $\frac{h^{n+7}}{h^4}$.
12. $\frac{6m^4n^6p^p}{2mn^2p^3}$.
13. $\frac{12a^4b^2c}{8ab^3c^2}$.
14. $(3a^3b^2c)^2$.
15. $(-3m^2n^3)^3$.
16. $\left(\frac{4ab}{m^2n}\right)^2$.
17. $\left(\frac{-2m}{5xy}\right)^2$.
18. $a^7 \cdot a^{x+4}$.
19. $(-3)^2 \cdot (-3)^3$.
20. $\frac{a^{x+4}b^6}{a^2b^6}$.
21. $\frac{a^{3n}x^{n+1}}{a^nx^3}$.
22. $(2a^{2n})^3$.
23. $(a^2b^3)^n$.
24. $(r^ns^{2n})^4$.
25. $(r^ns^{2n})^n$.
26. $\frac{a^{4n}b^m}{a^nb^3}$.
27. $\frac{a^{x+2}}{a^{x-2}}$.
28. $\frac{(a+b)^3}{a^3+b^3}$.
29. $\left(\frac{6a}{b}\right)^2 \cdot \left(\frac{b}{2a}\right)^3 \cdot \left(\frac{b}{3a}\right)^2$.
30. $\left(\frac{a^2}{x^3}\right)^n \cdot \left(\frac{c^2}{y^3}\right)^n \cdot \left(\frac{x^3y^3}{ac^2}\right)^n$.
31. $\frac{(x^4-y^4)^2}{(x^2+y^2)^3} \cdot \frac{x^8-y^8}{(x^4+y^4)^2}$.
32. $\frac{6a^{2x-3y}b^{3y-5}}{(2a^{x-2y}b^{y-9})^3}$.
33. $\frac{(a-1)^{3n}}{(1-a)^{2n}}$.
34. $\frac{(6a^2x^3y)^2}{(4ax^2y^2)^3}$.
35. $(1\frac{1}{3})^3 \cdot (1\frac{1}{2})^3 \cdot (\frac{1}{2})^3$.
36. Find the value of $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}$ for $x = 10$.
37. Find the value of $3x + 5 + \frac{4}{x} + \frac{9}{x^2} + \frac{2}{x^3}$ for $x = 10$.
38. Establish the laws of exponents II, III, IV, V, VI stating at each step the principle used.
39. State the six laws of exponents in words.

35. Meaning of $a^{\frac{1}{q}}$. The proofs of the above six laws of exponents assume that the exponents are positive integers. According to the definition of a^x (Art. 34), such an expression as $a^{\frac{1}{3}}$ has no meaning whatever. If we use such expressions, we must first give them a meaning. It is convenient to define them in such a way that the laws for positive integral exponents hold also for fractional exponents.

Assuming Law I, Art. 34, to hold, we shall have

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

Assuming that some number exists whose third power is a, we shall denote it by $a^{\frac{1}{3}}$. Another way of writing $a^{\frac{1}{3}}$ is $\sqrt[3]{a}$, which is read "the cube root of a." In general, if q is a positive integer,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots \text{ to } q \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+ \cdots \text{ to } q \text{ terms}} = a,$$

and $a^{\frac{1}{q}}$ means a number whose qth power is a. Another way of writing $a^{\frac{1}{q}}$ is $\sqrt[q]{a}$, which is read "the qth root of a."

Thus, the fractional exponent $\frac{1}{q}$ serves the same purpose as the radical sign $\sqrt[q]{}$.

36. Meaning of $a^{\frac{p}{q}}$. According to Law I, Art. 34, if p and q are positive integers,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots \text{ to } p \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+ \cdots \text{ to } p \text{ terms}} = a^{\frac{p}{q}},$$

and $a^{\frac{p}{q}}$ means **the pth power of the qth root of a.**

That is, $$a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

37. Principal roots. It will be seen later (Art. 102) that any number a has q distinct qth roots. Thus, the number 4 has the two square roots ± 2. We shall, however, for the present, consider only the arithmetical or positive value of $a^{\frac{1}{q}}$ when a is a positive number. With this limitation, it turns out that $a^{\frac{1}{q}}$ $(a > 0)$ has one and only one value. This value is often called the **principal qth root** of a. For example, $9^{\frac{1}{2}} = 3$, and not ± 3. Likewise, $\sqrt[4]{81} = 3$, and not ± 3. If both the positive and negative roots are meant, we shall write both signs before the radical.

Without this limitation, it will be seen from the following illustration that the pth power of a qth root of a number is not necessarily equal to a given qth root of the pth power:

$(4^{\frac{1}{2}})^4 = 16$, while the square root of 4^4 may be either $+16$ or -16.

38. Meaning of a^0. In order that the first law of exponents may hold for an exponent zero, it is necessary that

$$a^0 \cdot a^m = a^{0+m} = a^m, \qquad (1)$$

or,

$$a^0 = 1, \text{ if } a \neq 0.$$

That is, *any number a with the exponent* 0 *is equal to* 1, *provided* $a \neq 0$.

39. Meaning of a^n when n is negative. Let $n = -m$, where m is a positive number. By Law I, Art. 34, and Art. 38,

$$a^m \cdot a^{-m} = a^{m-m} = a^0 = 1. \qquad (1)$$

Hence,

$$a^{-m} = \frac{1}{a^m}, \text{ if } a \neq 0,$$

and

$$a^n = \frac{1}{a^{-n}}. \qquad (2)$$

That is, *by the use of formula* (2) *an expression containing negative exponents may be reduced to one containing only positive exponents.*

Illustration 1. $\dfrac{2^2 \cdot 3^{-4}}{7 \cdot 5^{-2}} = \dfrac{2^2 \cdot 5^2}{7 \cdot 3^4} = \dfrac{2^2}{7 \cdot 5^{-2} \cdot 3^4} = 2^2 \cdot 7^{-1} \cdot 3^{-4} \cdot 5^2.$

Illustration 2. The decimal 0.000,000,000,000,000,000,000,00166 gram gives the mass of the hydrogen atom which is more compactly written as $166(10)^{-26}$.

We have now found meanings for fractional, zero, and negative exponents consistent with the first law of exponents. To give logical completeness, it is necessary to show that the meanings are consistent * with all the laws of exponents, but we shall assume this.

EXERCISES

Oral Practice in the Use of Fractional, Negative, and Zero Exponents

Simplify the following expressions:

1. $(64)^{\frac{1}{2}}$.
2. $4x^0$.
3. $\dfrac{x^2 \cdot x^{-2}}{2^{-2}}$.
4. $\dfrac{6}{3^{-1}}$.
5. $\left(\dfrac{1}{3}\right)^{-3}$.
6. $1^n \cdot x^n \cdot x^{-n}$.
7. $(x^n \cdot x^{-n})^n$.
8. $(5a + 8)^0$.
9. $9^{\frac{3}{2}}$.
10. $(x^2)^{\frac{5}{2}}$.
11. $36^{\frac{1}{2}} \cdot 9^{-\frac{1}{2}}$.
12. $\left(\dfrac{1}{49}\right)^{-\frac{1}{2}}$.
13. $x^{\frac{1}{2}} \cdot x^{\frac{3}{2}} \cdot x^{-1}$.
14. $(32)^{\frac{1}{5}} \cdot x^{-1} \cdot x^2 \cdot x$.
15. $(0.25)^{\frac{1}{2}}$.
16. $(0.25)^{-\frac{1}{2}}$.
17. $(2^{-2} + 3^{-3})^0$.
18. $\left(-\dfrac{1}{8}\right)^{-1}$.

* See Chrystal's *Algebra*, Fifth edition, Part I, p. 182.

MISCELLANEOUS EXERCISES

Obtain expressions free from negative exponents equal to each of the following:

1. 4^{-2}.

2. $6^{-1} \cdot 3^2$.

3. $3^{-1} \cdot 3^{-2}$.

4. $7^0 \cdot 4^{-2}$.

5. $(-8)^{-\frac{1}{3}}$.

6. $x^{-3} \cdot x^{-4}$.

7. $\dfrac{x^4}{x^{-3}}$.

8. $(9^{-1})^{\frac{1}{2}}$.

9. $(4^{-1})^{-\frac{1}{2}}$.

10. $(16)^{-\frac{1}{2}}$.

11. $\left(\dfrac{25}{9}\right)^{-\frac{1}{2}}$.

12. $\left(\dfrac{9}{25}\right)^{-\frac{1}{2}}$.

13. $[(0.81)^{-1}]^{\frac{1}{2}}$.

14. $[(0.0036)^{-\frac{1}{2}}]^{-1}$.

15. $(.0025)^{-\frac{1}{2}}$.

16. $\dfrac{a^0y^{-1}}{y^{-2}}$.

17. $4x^{-2}y^{-4}y^6$.

18. $(x+y)^{-1}$.

Write the following without denominators by the use of negative exponents.

19. $\dfrac{x}{y^3}$. **20.** $\dfrac{8y}{z^4}$. **21.** $\dfrac{1}{(1.08)^{20}}$. **22.** $\dfrac{3x}{(1.05)^{30}}$. **23.** $\dfrac{3x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$.

Perform the indicated operations and simplify.

24. $(x^{\frac{1}{4}})^2$.

25. $(a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{2}})^6$.

26. $(32x^5y^{10})^{\frac{1}{5}}$.

27. $\left(\dfrac{x^{0.3}}{y^{0.7}}\right)^{0.2}$.

28. $(-a^6y^3z^{-9})^{\frac{7}{3}}$.

29. $(a^0x^{\frac{1}{2}}y^{-1})^{-2}$.

30. $\left(\dfrac{P^{1.42}}{P}\right)^{\frac{1}{3}}$.

31. $\left(\dfrac{P^{1.41}}{q}\right)^{\frac{1}{3}}$.

Multiply:

32. $(x^{\frac{5}{2}} + y^{\frac{5}{2}})$ by $(x^{\frac{5}{2}} - y^{\frac{5}{2}})$.

33. $x^{\frac{1}{3}}y^{\frac{2}{3}} + \dfrac{1}{x^{\frac{1}{3}}y^{\frac{2}{3}}}$ by $x^{\frac{2}{3}}y^{\frac{4}{3}}$.

34. $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$.

35. $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1$ by $a^{\frac{1}{3}} + 1$.

Divide:

36. $x - 1$ by $x^{\frac{1}{3}} - 1$.

37. $m^{-3} + 1$ by $m^{-1} + 1$.

38. $x^{\frac{5}{2}} + x^2 + x^{\frac{3}{2}} + x + x^{\frac{1}{2}} + x^0$ by $x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$.

Change the following expressions into equal expressions having as small a positive integer as possible with a fractional exponent.

39. $8^{\frac{1}{2}}$.

Solution: $8^{\frac{1}{2}} = (4 \cdot 2)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \cdot 2^{\frac{1}{2}}$.

40. $(32)^{\frac{1}{2}}$.

41. $(48)^{\frac{1}{4}}$.

42. $(18)^{\frac{1}{2}}$.

43. $(125)^{\frac{1}{2}}$.

44. $(81)^{\frac{1}{3}}$.

45. $(800)^{\frac{1}{2}}$.

46. $(2187)^{\frac{1}{6}}$.

Introduce the coefficients of the following parentheses into the parentheses.

47. $2(3)^{\frac{1}{2}}$.

Solution: $2(3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = (4 \cdot 3)^{\frac{1}{2}} = (12)^{\frac{1}{2}}$.

48. $3(11)^{\frac{1}{2}}$.

49. $2(5)^{\frac{1}{3}}$.

50. $2(7)^{\frac{1}{4}}$.

51. $3(13)^{\frac{1}{3}}$.

52. $7(6)^{\frac{1}{3}}$.

53. $4(3)^{\frac{1}{3}}$.

Simplify the following:

54. $\left(\dfrac{a^{n+1}}{a^n}\right)^n$.

55. $(x^{n-1} \cdot x^{n+1})^{\frac{1}{n}}$.

56. $\left(\dfrac{1}{a^{n-1}} \cdot \dfrac{1}{a^{n+1}}\right)^{\frac{1}{n}}$.

57. $\left[x(x \cdot x^{\frac{1}{2}})^{\frac{1}{2}}\right]^{\frac{1}{2}}$.

58. $(a^{2n} \cdot a^n b^{3n})^{\frac{1}{n}}$.

59. $\left(a^{2n} \cdot a^n b^{\frac{1}{3n}}\right)^{3n}$.

60. Find the value of

$$y = x^3 + 4x^2 + 3x - 8 \cdot 5^{\frac{1}{2}}$$

when $$x = 5^{\frac{1}{2}}.$$

61. If $pv^{\frac{3}{2}} = 10000$, calculate p when $v = 4$.

62. If $t = \dfrac{(2s)^{\frac{1}{2}}}{g^{\frac{1}{2}}}$, find t when $s = 100$, $g = 32$.

63. The number of revolutions per minute of a water wheel is given by the formula

$$n = 53.25\frac{F^{\frac{5}{4}}}{H^{\frac{1}{2}}},$$

where F is the fall of the water in feet, H the horsepower. Calculate n when $F = 16$, $H = 100$.

64. We may write 0.000,016 in the form $16 \cdot 10^{-6}$. Write the following numbers in briefer form by the use of negative exponents: 0.000,002, 0.000,000,004, 0.000,001,7.

65. The number of molecules in a pint of air under standard atmospheric conditions at 0° C is about 14,000,000,000,000,000,000,000. Write in briefer form by the use of 10 with an exponent.

66. The velocity of light is 29,986,000,000 centimeters per second. Write in briefer form by the use of 10 with an exponent.

67. Calculate the following differences.

$$\left[\left(\frac{1}{2}\right)^{-2} - (2)^{-2}\right]; \ [-(2)^{-2} - (-2)^{-2}]; \ \left[\left(\frac{1}{2}\right)^{-2} - 2^2\right]; \ \left[\left(\frac{1}{2}\right)^{-2} - (-2)^{-2}\right].$$

68. If $3^{\frac{1}{2}}$ and $2^{\frac{1}{2}}$ are substituted for x in the expression

$$x^5 - 2x^4 - 5x^3 + 10x^2 + 6x,$$

show that the results reduce to the same number.

69. Two spherical particles each one gram in mass whose centers are one centimeter apart attract each other with a force of 0.000,000,066,6 dyne. Express this number as an integer multiplied by 10 with an exponent.

70. Some authorities say that the mass of a hydrogen atom is 1.663×10^{-24} grams. How would this number be written in ordinary decimal notation?

71. The radius of the first Bohr ring of hydrogen is given as 0.5305×10^{-8} centimeters. Write this number in the ordinary decimal notation.

72. By the use of negative exponents express a micron as a part of a meter. (A micron is one millionth of a meter.)

73. The numbers 6867, 5896, 4861, 3934 each multiplied by 10^{-10} give the wave lengths in meters of deep red, yellow, blue, and ultraviolet, light, respectively. Express each wave length in microns (see exercise 72).

74. Show that $\dfrac{\frac{1}{3}y^{-\frac{1}{3}} + \frac{1}{3}y^{\frac{1}{3}}x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$ reduces to $\dfrac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$, when $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

75. Show that $\dfrac{\frac{1}{2} + \frac{1}{2}y^{\frac{1}{2}}x^{-\frac{1}{2}}}{x}$ reduces to $\dfrac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$, if $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

40. Radicals. An indicated root of a number is called a **radical.** Thus, $\sqrt{3}$, $\sqrt[3]{64}$, $\sqrt{\sqrt[4]{7}}$, and $\sqrt[11]{a+b}$ are radicals.

The **radicand** is the number of which the root is to be taken.

The number that tells what root is to be taken is called the **index** of the root.

Thus in $\sqrt{3}$, $6\sqrt[4]{19}$, and $8\sqrt[3]{x+y}$, the radicands are 3, 19, and $x + y$, and the indices are 2, 4, and 3, respectively.

The **order** of a radical is given by the index of the root.

Thus $\sqrt{5}$ is of the second order, $\sqrt[3]{85}$ is of the third order, and so on.

As stated in Art. 35, the nth root of a, written $\sqrt[n]{a}$, has the same meaning as $a^{\frac{1}{n}}$. That is, $\sqrt[n]{a}$ means a number whose nth power, $(\sqrt[n]{a})^n$, is a. Thus, by definition, $(\sqrt[4]{5})^4 = 5$.

ORAL EXERCISES

1. Give the index of each of the radicals $\sqrt{6}$, $\sqrt[3]{a}$, $\sqrt[7]{a+b}$.
2. Give the order of each of the radicals $\sqrt{6}$, $\sqrt[3]{a}$, $\sqrt[7]{a+b}$.
3. What is the radicand in each of the radicals $\sqrt{6}$, $\sqrt[3]{a}$, $\sqrt[7]{a+b}$.
4. Fill the blanks in $(\sqrt{3})^2 =$ ——, $(\sqrt[5]{7})^5 =$ ——, $(\sqrt[8]{27})^8 =$ ——.
5. Express $\sqrt[3]{a^2}$ by means of fractional exponents.

41. Changes in the form of a radical. Changes in the form of a radical may often be made by use of the definition of the nth root of a number, say of a, which implies that $(\sqrt[n]{a})^n = a$, and by the use of the following equalities:

$$\text{I.} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

$$\text{II.} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

$$\text{III.} \quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

The truth of these equalities is apparent from the laws of exponents. Thus,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab},$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{a}{b}},$$

$$\sqrt[m]{\sqrt[n]{a}} = \left(a^{\frac{1}{n}}\right)^{\frac{1}{m}} = a^{\frac{1}{mn}} = \sqrt[mn]{a}.$$

Oral Exercise. In so far as possible, state I, II, and III in words.

Radicals are frequently changed to advantage in one or more of the following ways:

(1) By removing factors from the radicand.

Illustration 1. $\sqrt{75} = \sqrt{5^2 \cdot 3} = \sqrt{5^2} \cdot \sqrt{3} = 5\sqrt{3}.$

Illustration 2. $\sqrt[3]{24a^4b^3} = \sqrt[3]{2^3 \cdot 3a^4b^3} = \sqrt[3]{2^3a^3b^3} \cdot \sqrt[3]{3a} = 2ab\sqrt[3]{3a}.$

(2) By introducing a coefficient under the radical sign.

Illustration 1. $7\sqrt{2} = \sqrt{7^2} \cdot \sqrt{2} = \sqrt{7^2 \cdot 2} = \sqrt{98}.$

Illustration 2. $2a^2b\sqrt[3]{abx} = \sqrt[3]{8a^6b^3} \cdot \sqrt[3]{abx} = \sqrt[3]{8a^7b^4x}.$

(3) By reducing a radical with a fractional radicand to one whose radicand is integral.

Illustration 1. $\sqrt{\frac{2}{5}} = \sqrt{\frac{2}{5} \cdot \frac{5}{5}} = \frac{\sqrt{10}}{\sqrt{25}} = \frac{\sqrt{10}}{5} = \frac{1}{5}\sqrt{10}.$

Illustration 2. $\sqrt{\frac{a}{b}} = \sqrt{\frac{a}{b} \cdot \frac{b}{b}} = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b}.$

This process is called **rationalizing the denominator** as no radical remains in the denominator.

(4) By reducing a radical to lower order.

Illustration 1. $\sqrt[4]{100} = \sqrt[4]{2^2 \cdot 5^2} = \sqrt{\sqrt{2^2 \cdot 5^2}} = \sqrt{2 \cdot 5} = \sqrt{10}.$

Exercise. Carry out an equivalent process in fractional exponents for $(100)^{\frac{1}{4}}$.

Illustration 2. $\sqrt[6]{8a^3b^{3n}} = (2^3 \cdot a^3b^{3n})^{\frac{1}{6}} = 2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{n}{2}} = \sqrt{2ab^n}.$

A radical is said to be in its **simplest form**: (1) when the radicand contains no factor to a power whose exponent equals the order of the radical, (2) when the radicand is integral, (3) when the order of the radical is as small as possible.

EXERCISES

Change each of the following to a radical form:

1. $a^{\frac{2}{3}}$.
2. $3x^{\frac{3}{2}}$.
3. $2^{\frac{3}{2}}x^{\frac{1}{2}}$.
4. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.
5. $(x+y)^{\frac{6}{7}}$.
6. $ax^{\frac{2}{3}}$.
7. $(pq)^{\frac{1}{2}}$.
8. $x^{\frac{5}{2}}yz^{\frac{3}{2}}$.
9. $\left(\frac{1}{2}\right)^{\frac{1}{3}}$.

Change each of the following to a form involving only exponents instead of radicals:

10. $\sqrt{a}$.
11. $\sqrt[4]{a^3}$.
12. $\sqrt{a^2b^6c^4}$.
13. $\sqrt{a^3b^5c^4}$.
14. $\sqrt[3]{a}\cdot\sqrt{b^6}$.
15. $\sqrt[n]{a^nb^{2n}c^{3n}}$.
16. $\sqrt[3]{a^6b^2c}$.
17. $\sqrt[4]{a+b}$.
18. $\sqrt[5]{\dfrac{x^{10}y^5}{m^{15}n^{30}}}$.
19. $\sqrt[3]{\dfrac{ab^2c^3}{x^5yz^4}}$.
20. $\sqrt{a^{\frac{3}{4}}b}\cdot\sqrt[3]{x^2}$.
21. $\sqrt[3]{a^{-6}b^{-2}}$.
22. $\sqrt{a^{-2q}b^{2q}}$.
23. $\sqrt[3]{\sqrt{a^5}}$.
24. $\sqrt[4]{16a^{-8}c^{-3}}$.

Introduce the coefficient of each of following under the radical sign:

25. $5\sqrt{3}$.
26. $2\sqrt[3]{2}$.
27. $6\sqrt{7}$.
28. $3\sqrt[3]{5}$.
29. $10\sqrt[5]{.01}$.
30. $a\sqrt{b}$.
31. $-5mn\sqrt[3]{3m^2}$.
32. $\frac{1}{2}\sqrt{6}$.
33. $\frac{3}{5}\sqrt[3]{25am}$.
34. $2\sqrt{a+b}$.
35. $(1+x)\sqrt{1+x}$.
36. $(a+b)\sqrt[3]{3}$.

Change each of the following to a radical of lower order:

37. $\sqrt[6]{81}$.

Solution: $\sqrt[6]{81} = \sqrt[3]{\sqrt{81}} = \sqrt[3]{9}$.

38. $\sqrt[4]{4}$.
39. $\sqrt[4]{100}$.
40. $\sqrt[6]{216}$.
41. Perform the operations in exercises 37–40 using fractional exponents.

Change the following to radicals of the same order:

42. $\sqrt{a}$, $\sqrt[3]{a}$.

Solution:

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}.$$
$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

43. $2\sqrt{xy}$, $5\sqrt[3]{xy^2z^2}$.

Solution:

$$2\sqrt{xy} = 2x^{\frac{1}{2}}y^{\frac{1}{2}} = 2x^{\frac{3}{6}}y^{\frac{3}{6}} = 2\sqrt[6]{x^3y^3}.$$
$$5\sqrt[3]{xy^2z^2} = 5x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}} = 5x^{\frac{2}{6}}y^{\frac{4}{6}}z^{\frac{4}{6}} = 5\sqrt[6]{x^2y^4z^4}.$$

44. $\sqrt{2}$, $\sqrt[3]{4}$.
45. $\sqrt[3]{b}$, $\sqrt[4]{c}$.
46. $\sqrt{xy}$, $2\sqrt[4]{mn}$.
47. $\sqrt[3]{5}$, $\sqrt[5]{7}$.
48. $\sqrt{ab}$, $\sqrt[3]{x^2y}$.
49. $2\sqrt{m}$, $\sqrt[6]{x^2y^5}$.
50. $2\sqrt{2}$, $3\sqrt[3]{3}$, $5\sqrt[5]{5}$.
51. $\sqrt[3]{x^2y^2z}$, $2\sqrt[4]{xy^3z^2}$.
52. $\frac{1}{2}\sqrt[5]{ab^3}$, $4\sqrt{ab^5c^3}$

Reduce each radical to simplest form:

53. $\sqrt{8}$.
54. $\sqrt{27}$.
55. $\sqrt{12a^2b^3}$.
56. $\sqrt{1800}$.
57. $\sqrt[3]{54}$.
58. $\sqrt{\frac{2}{3}}$.
59. $\sqrt{125}$.
60. $\sqrt{0.125}$.
61. $\sqrt{\dfrac{8x^4}{27}}$.
62. $\sqrt{\frac{1}{2}}$.
63. $\sqrt{6\frac{1}{8}}$.
64. $\sqrt[3]{-\frac{1}{8}}$.
65. $\sqrt[4]{144}$.
66. $\sqrt{3\sqrt[3]{3}}$.
67. $\sqrt[4]{a^2 - 2ab + b^2}$.
68. $\sqrt[6]{(a^2 + b^2)^2}$.
69. $\sqrt[3]{-\frac{16}{54}}$.
70. $\sqrt[6]{\dfrac{a^2}{16c^{10}}}$.
71. $\sqrt{28a^6b}$.
72. $\sqrt[3]{\dfrac{72x^2y^6}{z^6}}$.
73. $\sqrt{49a^{2n}b^8c^4}$.

42. Rational and irrational numbers. A **rational number** is defined as one that can be expressed as the quotient of two integers. A real number which cannot be thus expressed is called an **irrational number.**

Thus, 16, $\frac{1}{2}$, and $\frac{19}{23}$ are rational numbers; $\sqrt{2}$,* $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt{5}$, $1 + \sqrt{5}$, and $9^{\frac{1}{3}}$ are irrational numbers.

Any irrational number can be inclosed between two rational numbers that differ from one another by as small a number as we please.

Thus, we may write,

$$1 < \sqrt{2} < 2.$$
$$1.4 < \sqrt{2} < 1.5.$$
$$1.41 < \sqrt{2} < 1.42.$$

* To show that $\sqrt{2}$ cannot be expressed as the quotient of two integers, suppose it is possible that

$$\sqrt{2} = \frac{m}{n},$$

where $\frac{m}{n}$ is a rational fraction in its lowest terms. At least one of the numbers m or n is odd. Clearing of fractions and squaring both sides, we get

$$2n^2 = m^2.$$

From this equation, we see that m^2 is an even number. Hence m is an even number. If m is even, m^2 contains the factor 4. Hence n^2 is an even number, and n is itself even. This is contrary to hypothesis that $\frac{m}{n}$ is a fraction in its lowest terms.

This proof is found in Euclid (about 300 B.C.), and is supposed to be due to a much earlier mathematician than Euclid.

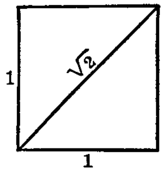

FIG. 10

Either of the two sequences of numbers in the two outer columns determines $\sqrt{2}$ in much the same way that .3 .33, .333, . . . determine $\frac{1}{3}$.

As a geometrical illustration of an irrational number we may take the diagonal of a square whose side is 1.

ORAL EXERCISES

1. Tell which of the following numbers are rational: $5, \frac{6}{0.3}, \sqrt{64}, 0.5, \sqrt{10}$, $0.444, \sqrt{2}+1$.
2. Give an example of a rational number that is not an integer.
3. Can every integer be expressed as the quotient of two integers? Explain.

43. Addition and subtraction of radicals. Two radicals which have the same order and the same radicand are said to be **similar.**

Thus, $5\sqrt{2}$ and $-2\sqrt{2}$ are similar; so also are $\sqrt[3]{81}$ and $2b\sqrt[3]{3a^3}$ since $\sqrt[3]{81} = 3\sqrt[3]{3}$ and $2b\sqrt[3]{3a^3} = 2ab\sqrt[3]{3}$.

On the other hand, $\sqrt{2}$ and $\sqrt{3}$ are dissimilar; so are $\sqrt{2}$ and $\sqrt[3]{2}$.

The algebraic sum of similar radicals equals the common radical factor multiplied by the sum of its coefficients.

Illustration 1. $8\sqrt{2} + 3\sqrt{2} - 4\sqrt{2} = (8+3-4)\sqrt{2} = 7\sqrt{2}$.

Illustration 2.
$$\begin{aligned}\sqrt{75} + 3\sqrt{12} - 5\sqrt{27} &= \sqrt{25\cdot 3} + 3\sqrt{4\cdot 3} - 5\sqrt{9\cdot 3}\\ &= 5\sqrt{3} + 6\sqrt{3} - 15\sqrt{3}\\ &= -4\sqrt{3}.\end{aligned}$$

Illustration 3.
$$\begin{aligned}&\frac{1}{2}\sqrt{a^3} + 2b\sqrt{\frac{b^2}{a}} - 3b\sqrt{25a^5b^2}\\ &= \frac{1}{2}\sqrt{a^2\cdot a} + 2b\sqrt{\frac{ab^2}{a^2}} - 3b\sqrt{25a^4b^2a}\\ &= \frac{a}{2}\sqrt{a} + \frac{2b^2}{a}\sqrt{a} - 15a^2b^2\sqrt{a}\\ &= \left(\frac{a}{2} + \frac{2b^2}{a} - 15a^2b^2\right)\sqrt{a}.\end{aligned}$$

EXERCISES

Perform the indicated operations and simplify:

1. $4\sqrt{2} + 8\sqrt{2}$.
2. $\sqrt{3} - 2\sqrt{3} + 9\sqrt{3}$.
3. $\sqrt{20} + 8\sqrt{45} - 2\sqrt{5}$.
4. $3\sqrt{28} - \sqrt{63} + 4\sqrt{175}$.
5. $\sqrt[3]{81} + 5\sqrt[3]{24} - \sqrt[3]{375}$.
6. $\sqrt[3]{16} + 9\sqrt[3]{250}$.
7. $6\sqrt{\frac{3}{8}} - \sqrt{24} - \sqrt{\frac{2}{3}} + 8\sqrt{6}$.
8. $\sqrt{a} + 6\sqrt{a} - 2\sqrt{a}$.
9. $3\sqrt{b^3} + 4\sqrt{a^2bc^4} + \sqrt{4b^5c^2}$.
10. $\sqrt[3]{a^2b} + 4\sqrt[3]{8a^5b^4c^3} + 3\sqrt[3]{64a^2b^7}$.

11. $\sqrt[4]{a^4bc^6} + 8\sqrt[4]{b^9c^2} - 5\sqrt[4]{a^8b^5c^2}$.

12. $\sqrt[3]{\frac{1}{b}} + \sqrt[3]{a^3b^2} - \sqrt[3]{27b^5}$.

13. $\sqrt{(a+b)^3} - \sqrt{a^3 + a^2b} - \sqrt{ab^2 + b^3}$.

14. $\sqrt{a} + \sqrt{\frac{1}{a}} + \sqrt{a^3 + 2a^2 + a}$.

15. $\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}$.

16. $3a^{\frac{1}{2}}b + 4a^{\frac{3}{2}}c + 5a^{\frac{5}{2}}d$.

17. $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 5x^{\frac{3}{2}}y^{\frac{5}{2}} + 2x^{\frac{1}{2}}y^{\frac{3}{2}}$.

Solution: Taking out the common monomial factor $x^{\frac{1}{2}}y^{\frac{1}{2}}$, we have

$$2x^{\frac{1}{2}}y^{\frac{1}{2}} - 5x^{\frac{3}{2}}y^{\frac{5}{2}} + 2x^{\frac{1}{2}}y^{\frac{3}{2}} = (2 - 5xy^2 + 2y)x^{\frac{1}{2}}y^{\frac{1}{2}}.$$

18. $3a^{\frac{1}{3}}y^{\frac{1}{3}} - 2a^{\frac{7}{3}}y^{\frac{2}{3}}$.

19. $2^{\frac{1}{2}}a^{\frac{1}{2}}b + 8^{\frac{1}{2}}a^{\frac{3}{2}}b^2 - 18^{\frac{1}{2}}a^{\frac{1}{2}}b^3$.

20. $t^{\frac{1}{2}} + 2t^{\frac{3}{2}} + 3t^{\frac{5}{2}}$.

21. $3^{\frac{1}{2}} + 12^{\frac{1}{2}}a - 27^{\frac{1}{2}}b$.

22. $\sqrt{a^3b^2} + 3\sqrt{4a^3b^4} + 5a\sqrt{9ab^2} - 4ab\sqrt{ab^2}$.

23. $3\sqrt{8} - 4\sqrt{72} + 6\sqrt{48} - \sqrt{108}$.

24. $\sqrt[3]{(x+y)^4} - \sqrt[3]{x^4 + x^3y} - \sqrt[3]{8xy^3 + 8y^4}$.

44. Multiplication of radicals. The product of two radicals is obtained by use of the principle $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. (Art. 41.) We shall illustrate the process of finding the product of two radicals by examples.

Illustration 1. Multiply $\sqrt{3}$ by $\sqrt{5}$.

Solution:
$$\sqrt{3} \cdot \sqrt{5} = 3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 15^{\frac{1}{2}} \quad \text{(Law III, Art. 34.)}$$
$$= \sqrt{15}.$$

Illustration 2. Multiply $2\sqrt[3]{ab^2m}$ by $3\sqrt[3]{mn}$.

Solution:
$$2\sqrt[3]{ab^2m} \cdot 3\sqrt[3]{mn} = 2(ab^2m)^{\frac{1}{3}} \cdot 3(mn)^{\frac{1}{3}}.$$
$$= 6(ab^2m^2n)^{\frac{1}{3}} \quad \text{(Law III, Art. 34.)}$$
$$= 6\sqrt[3]{ab^2m^2n}.$$

Radicals of different orders may be reduced to the same order by the methods of Art. 41.

Illustration 3. Multiply $2\sqrt{ab}$ by $5\sqrt[3]{a^2b}$.

Solution:
$$2\sqrt{ab} = 2\sqrt[6]{a^3b^3},$$
$$5\sqrt[3]{a^2b} = 5\sqrt[6]{a^4b^2},$$
$$2\sqrt[6]{a^3b^3} \cdot 5\sqrt[6]{a^4b^2} = 10\sqrt[6]{a^7b^5} = 10a\sqrt[6]{ab^5}.$$

EXERCISES

Perform indicated multiplications and simplify as far as possible:

1. $3\sqrt{m} \cdot 5\sqrt{an}$.
2. $4\sqrt{abc} \cdot a\sqrt{bc}$.
3. $2\sqrt{5} \cdot \sqrt{10} \cdot 4\sqrt{35}$.
4. $\sqrt{x} \cdot \sqrt[3]{y^2}$.
5. $a\sqrt{b} \cdot \sqrt[3]{ab^2}$.
6. $\sqrt{10} \cdot \sqrt[4]{2}$.
7. $2\sqrt{a} \cdot 5\sqrt[3]{a^2b} \cdot \sqrt[4]{a^3c}$.
8. $\sqrt[5]{4} \cdot \sqrt[5]{8}$.
9. $\sqrt[3]{6} \cdot \sqrt[3]{36} \cdot \sqrt[5]{5}$.
10. $\sqrt{5} \cdot \sqrt[3]{\frac{2}{5}}$.

The multiplication of radicals is often much more easily performed by the use of fractional exponents. That method should be used in the next ten exercises.

11. $\sqrt[3]{m} \cdot \sqrt[3]{m^2n} \cdot 5\sqrt[4]{m^3n^2}$.

Solution:
$$\begin{aligned}\sqrt[3]{m} \cdot \sqrt[3]{m^2n} \cdot 5\sqrt[4]{m^3n^2} &= m^{\frac{1}{3}}m^{\frac{2}{3}}n^{\frac{1}{3}}5m^{\frac{3}{4}}n^{\frac{1}{2}} \\ &= 5m^{\frac{7}{4}}n^{\frac{5}{6}} = 5m \cdot m^{\frac{3}{4}}n^{\frac{5}{6}} = 5m \cdot m^{\frac{9}{12}}n^{\frac{10}{12}} \\ &= 5m\sqrt[12]{m^9n^{10}}.\end{aligned}$$

12. $\sqrt{xy} \cdot 2\sqrt[3]{x^2y^2}$.
13. $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt[4]{a^2b^3}$.
14. $\sqrt{3} \cdot \sqrt[3]{3}$.
15. $3\sqrt{5} \cdot \sqrt[4]{2}$.
16. $\sqrt[5]{128} \cdot \sqrt[3]{500}$.
17. $x^{\frac{4}{5}}y^{\frac{7}{10}}z^{\frac{3}{5}} \cdot xy^{\frac{2}{5}}z^{\frac{1}{4}} \cdot x^{\frac{3}{5}}z^{\frac{1}{5}}$.
18. $3a^{\frac{1}{2}}c^{\frac{1}{3}} \cdot 5a^{\frac{2}{3}}bc$.
19. $-8a^{\frac{3}{4}}b^{\frac{1}{6}} \cdot -5a^{\frac{3}{2}}bc^2$.
20. $\frac{2}{3}a^{\frac{2}{3}}b^2c^{\frac{1}{9}} \cdot 9a^{\frac{1}{5}}c^3$.
21. $-r^{\frac{1}{2}}s^{\frac{2}{3}} \cdot 3r^2sp$.
22. $(\sqrt{a} + 2\sqrt{b})(3\sqrt{a} - 5\sqrt{b})$.

Solution:
$$\begin{array}{r}\sqrt{a} + 2\sqrt{b} \\ 3\sqrt{a} - 5\sqrt{b} \\ \hline 3a + 6\sqrt{ab} \qquad\quad \\ -5\sqrt{ab} - 10b \\ \hline 3a + \sqrt{ab} - 10b.\end{array}$$

23. $(\sqrt{3} - \sqrt{5})(\sqrt{3} + \sqrt{5})$.
24. $(\sqrt{m} + \sqrt{n})^2$.
25. $2\sqrt{3}(\sqrt{3} + \sqrt{5} - 4\sqrt{6})$.
26. $2a\sqrt{b}(ab\sqrt{b} - a\sqrt{ab} + 5\sqrt{a})$.
27. $(5 - 2\sqrt{5})(3 - \sqrt{5})$.
28. $(\sqrt{7} + \sqrt{11})(\sqrt{3} - \sqrt{5})$.
29. $(\sqrt{2} + 2\sqrt{3} - 3\sqrt{5})^2$.
30. $(a^{\frac{1}{2}}b^{\frac{2}{3}} - c^{\frac{1}{4}})^2$.
31. $(3a^{\frac{1}{3}} + 2b^{\frac{1}{3}})^2$.
32. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$.
33. $(2\sqrt{7} - 8\sqrt{28} - \sqrt{63})^2$.
34. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
35. $(\sqrt{5} + 4\sqrt{6} - 2\sqrt{3})^2$.
36. Find the value of $x^2 - 4x + 1$ if $x = 2 + \sqrt{3}$.
37. Find the value of $3x^2 + 4x - 2$ if $x = \dfrac{-2 - \sqrt{10}}{3}$.
38. Find the value of $ax^2 + bx + c$ if $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$.

45. Division of radicals — rationalization of denominators. Division of radicals of the same order may be performed by the use of the principle

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}. \quad \text{(See Art. 41.)}$$

Illustration 1. $\dfrac{6\sqrt{6}}{2\sqrt{3}} = 3\sqrt{\dfrac{6}{3}} = 3\sqrt{2}.$

For purposes of computation it is usually desirable that the denominator of the quotient be made rational. In fact, division of radicals usually becomes mainly a process called **rationalizing the denominator.**

Illustration 2. $\dfrac{\sqrt{10}}{\sqrt{7}} = \dfrac{\sqrt{10}\,\sqrt{7}}{\sqrt{7}\,\sqrt{7}} = \dfrac{\sqrt{70}}{7}.$

Illustration 3. $\dfrac{\sqrt[3]{14}}{\sqrt[3]{36}} = \dfrac{\sqrt[3]{14}\cdot\sqrt[3]{6}}{\sqrt[3]{36}\,\sqrt[3]{6}} = \dfrac{\sqrt[3]{84}}{\sqrt[3]{216}} = \dfrac{\sqrt[3]{84}}{6}.$

Illustration 4. $\dfrac{3\sqrt{5}+\sqrt{2}}{2\sqrt{5}-3\sqrt{2}} = \dfrac{(3\sqrt{5}+\sqrt{2})\cdot(2\sqrt{5}+3\sqrt{2})}{(2\sqrt{5}-3\sqrt{2})\cdot(2\sqrt{5}+3\sqrt{2})}$

$$= \frac{30+2\sqrt{10}+9\sqrt{10}+6}{20-18} = \frac{36+11\sqrt{10}}{2}.$$

EXERCISES

Perform the following divisions, obtaining results with rational denominators:

1. $\sqrt{6} \div \sqrt{3}.$
2. $\sqrt{80} \div \sqrt{5}.$
3. $\sqrt{42} \div \sqrt{6}.$
4. $\sqrt[3]{35} \div \sqrt[3]{5}.$
5. $\sqrt[3]{135} \div \sqrt[3]{5}.$
6. $\sqrt{ab} \div \sqrt{a}.$
7. $\sqrt[3]{a^2bc^2} \div \sqrt[3]{ac^2}.$
8. $6\sqrt{5} \div 2.$
9. $6\sqrt{5} \div 2\sqrt{5}.$
10. $\sqrt{\frac{1}{2}} \div \sqrt{\frac{2}{3}}.$
11. $\sqrt[3]{ab} \div \sqrt[3]{cd}.$
12. $\sqrt[3]{a^2m} \div \sqrt[3]{am^2}.$
13. $\sqrt[3]{x} \div \sqrt[3]{y}.$
14. $6\sqrt{150} \div 5\sqrt{45}.$
15. $(\sqrt{12} - 4\sqrt{6}) \div \sqrt{2}.$
16. $(\sqrt{10} + 3\sqrt{15} - 7\sqrt{35}) \div \sqrt{10}.$
17. $6 \div 2\sqrt[3]{7}.$
18. $7 \div 2\sqrt[4]{5}.$
19. $\sqrt{\dfrac{a}{b}} \div \sqrt{ab}.$
20. $(\sqrt{a} + \sqrt{b}) \div \sqrt{ab}.$
21. $(\sqrt{a} + \sqrt{b}) \div \sqrt{a+b}.$
22. $(3 + \sqrt{6}) \div (2\sqrt{6} - 1).$
23. $1 \div (2\sqrt{3} - 4\sqrt{5}).$
24. $(5\sqrt{7} - 3\sqrt{11}) \div (\sqrt{7} + 4\sqrt{11}).$
25. $7\sqrt{2} \div (2\sqrt{5} - 9\sqrt{2}).$

26. $\dfrac{3}{3+\sqrt{6}}$.

27. $\dfrac{2}{2-2^{\frac{1}{2}}}$.

28. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

29. $\dfrac{5\sqrt{3}-3\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.

Given $2^{\frac{1}{2}} = 1.4142$, $3^{\frac{1}{2}} = 1.7321$, $5^{\frac{1}{2}} = 2.2361$, evaluate the expression in each of the following exercises to four significant figures * both before and after rationalizing the denominator.

Does it save time in each computation to rationalize the denominator?

30. $\dfrac{1}{\sqrt{3}}$.

31. $\dfrac{7}{\sqrt{5}-\sqrt{3}}$.

32. $\dfrac{2}{\sqrt{3}+1}$.

33. $\dfrac{3}{2^{\frac{1}{2}}+3^{\frac{1}{2}}}$.

34. $\dfrac{5^{\frac{1}{2}}+2^{\frac{1}{2}}}{5^{\frac{1}{2}}-2^{\frac{1}{2}}}$.

35. $\dfrac{1}{2+3^{\frac{1}{2}}}$.

36. $\dfrac{1}{\sqrt{5}}$.

37. $\dfrac{5}{\sqrt{5}+\sqrt{3}}$.

38. $\dfrac{4}{\sqrt{2}-\sqrt{3}}$.

46. Solution of equations containing radicals. Certain equations in which the unknown is involved under the radical sign can be reduced to equations of the first degree. The following examples illustrate the method of solving some of the more simple but typical equations in which such reduction can be made.

EXAMPLES

1. Solve the equation $\sqrt{3x+1} = 5$.

Solution: Squaring both members,

$$3x + 1 = 25.$$

Solving for x,

$$x = 8.$$

Check:

$$\sqrt{25} = 5.$$

It should be recalled that $\sqrt{25} = +5$, and does not equal ± 5; that is, when no sign precedes the radical the positive value of the root is to be taken. If both positive and negative roots are meant, we shall write both signs before the radical.

---

* In giving a result such as 2.2361 to four significant figures, we write 2.236. In giving the same result to three significant figures, we write 2.24 rather than 2.23, for 2.24 differs less from 2.236 than 2.23 differs from 2.236. In fact, it is usually desirable in giving any number of figures of an approximate result, to find whether the next figure beyond those to be retained in the result is less or greater than 5; for, if we should obtain a result 2.23 and know that the next figure > 5, the result should be given to three significant figures as 2.24.

2. Solve $\sqrt{4x+5} + 2\sqrt{x-3} = 17$.

Solution: Transposing,

$$\sqrt{4x+5} - 17 = -2\sqrt{x-3}.$$

Squaring, $\quad 4x + 5 - 34\sqrt{4x+5} + 289 = 4x - 12.$

Transposing and simplifying, $\quad \sqrt{4x+5} = 9.$

Squaring, $\quad 4x + 5 = 81.$

Solving for x, $\quad x = 19.$

Check: $\quad \sqrt{81} + 2\sqrt{16} = 17$

or, $\quad 17 = 17.$

3. Solve $(x-2)^{\frac{1}{2}} - (x+3)^{\frac{1}{2}} = 1$.

Solution: Transposing $(x+3)^{\frac{1}{2}}$ and squaring,

$$x - 2 = x + 3 + 2(x+3)^{\frac{1}{2}} + 1.$$

This reduces to $\quad (x+3)^{\frac{1}{2}} = -3.$

Squaring both sides and solving, $\quad x = 6.$

But 6 is not a solution of the given equation. In fact, the given equation has no solution.

Example 3 illustrates the fact that results obtained by squaring the sides of an equation containing radicals must be checked by substitution in the original equation to determine whether or not a result is a solution of that equation.

STEPS IN THE SOLUTION. In solving equations containing radicals it is usually convenient to proceed as follows:

(1) *Isolate the radical; that is, place it by itself on one side of the equation. If more than one radical occurs, isolate the most complicated one.*

(2) *Raise both sides of the equation to a suitable power.*

(3) *If a radical remains, isolate it and again raise to a suitable power.*

(4) *Solve the resulting equation.*

(5) *Check the result.*

EXERCISES

Solve and check by substitution.

1. $\sqrt{x+3} = 2.$

2. $x^{\frac{1}{2}} = 5.$

3. $\sqrt{3x-5} + 5 = 0.$

4. $1 + \sqrt{x-1} = 3.$

5. $\sqrt[3]{x+3} = 1.$

Hint: Cube each member.

6. $\sqrt[3]{x-4} + 7 = 9.$

7. $(2x-6)^{\frac{2}{3}} = 4.$

8. $10 - x = -(x^2 - 5)^{\frac{1}{2}}$.

9. $\sqrt{x + 15} = \sqrt{x + 1} + 2$.

10. $\sqrt{9x + 6} - 3\sqrt{x - 11} - 5 = 0$.

11. $\sqrt{x - 4} - \sqrt{2x - 10} = 0$.

12. $\dfrac{(8x + 4)^{\frac{1}{2}}}{(x + 5)^{\frac{1}{2}}} = 2$.

13. $\dfrac{(x^{\frac{1}{2}} + 2^{\frac{1}{2}})}{(x + 2)^{\frac{1}{2}}} = 1$.

14. Solve $V = \frac{4}{3}\pi r^3$ for r.

15. Solve $A = \dfrac{s^2\sqrt{3}}{4}$ for s.

16. Solve $u = \sqrt{v^2 + 2fs}$ for s.

17. Solve $A = P(1 + i)^t$ for i.

18. Solve $s = \dfrac{mgl}{\pi r^2 k}$ for r.

19. Solve $2 + \sqrt[3]{t - 5} = 13$ for t.

20. Solve $d = .02758\sqrt{D \cdot l \cdot \sqrt{P}}$ for P.

21. Using the formula $s = \dfrac{gt^2}{2}$, where

s = height in feet from which an object falls,
g = 32.2 feet per second,
t = time in seconds,

find how long it will take an object to fall to the earth from the top of the Washington Monument which is 555 feet high.

22. The velocity, v, of a falling body, starting from rest, is given by the formula $v = \sqrt{2gs}$.

(a) Solve for the distance s in terms of v and g.
(b) Calculate s if $g = 32.2$ and $v = 128.8$ feet per second.

47. Imaginary numbers. The square of any real number, positive or negative, is a positive number. The square root of a negative number cannot then be a real number and is given the name **imaginary number.** The imaginary number $\sqrt{-1}$ occurs so often that a special symbol is used for it; viz.

$$\sqrt{-1} = i.$$

Any imaginary number involves the product of a real number and i. For example,

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i,$$
$$\sqrt{-a} = \sqrt{a}\sqrt{-1} = i\sqrt{a}.$$

To perform operations with imaginary numbers, replace any such number, say $\sqrt{-a}$, by $i\sqrt{a}$, and operate with i as with any other letter, but replace i^2 in any expression by -1. Thus if a and b are real positive numbers

$$\sqrt{-a} \cdot \sqrt{-b} \neq \sqrt{a \cdot b},$$

but $\sqrt{-a} \cdot \sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{a}\sqrt{b} = -\sqrt{ab}.$

In Chapter XIII, the student will find a fuller discussion of imaginary numbers, and in more advanced mathematics and in practical applications he will find that they play a very important role, comparable to that of the real numbers.

EXERCISES

Express in terms of i.

1. $\sqrt{-64}$.
2. $-\sqrt{-81}$.
3. $-\sqrt{-20a^2}$.
4. $\sqrt{-\frac{9}{4}}$.
5. $\sqrt{-36b^2}$.
6. $\sqrt{-6a^2x^2}$.

Perform indicated operations and simplify when possible by replacing i^2 by -1.

7. $(1+i)(2-i)$.

Solution:
$$\begin{aligned}(1+i)(2-i) &= 2+i-i^2,\\ &= 2+i+1, \text{ since } i^2=-1,\\ &= 3+i.\end{aligned}$$

8. $(2-i)(2+i)$.
9. $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$.
10. $(-1+\sqrt{-3})^2$.
11. $(x+ai)(x-ai)$.
12. $\frac{i^3+1}{i+1}$.
13. $(2-\sqrt{-3})(2+\sqrt{-3})$.
14. $(24+7i)(24-7i)$.
15. $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}$.
16. $(1-i^2)(1+i^2)$.
17. $(4+3i)^2+(4-3i)^2$.
18. $(a+bi)^3+(a-bi)^3$.
19. $(3+4i)^4+(3-4i)^4$.
20. Find the value of x^3-8 when $x=-1+i\sqrt{3}$.
21. Find the value of x^2+x+1 when $x=\frac{-1-i\sqrt{3}}{2}$.

MISCELLANEOUS EXERCISES AND PROBLEMS

Perform the indicated operations and simplify when possible.

1. $\sqrt{3}\cdot\sqrt[3]{3}$.
2. $\sqrt[3]{2}\cdot\sqrt{2}$.
3. $\sqrt{\frac{1}{2}}\cdot\sqrt[3]{\frac{1}{2}}$.
4. $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt[4]{2}$.
5. $\sqrt{\frac{1}{3}}\cdot\sqrt[3]{3}$.
6. $3\sqrt{a}\cdot 7\sqrt[3]{b}$.
7. $(2\cdot 3^{\frac{1}{2}}+3\cdot 2^{\frac{1}{2}})(2^{\frac{1}{2}})$.
8. $(5^{\frac{1}{2}}+2)(5^{\frac{1}{2}}-2)$.
9. $(2^{\frac{1}{2}}+3^{\frac{1}{2}})^2$.
10. $(3\sqrt{2}-5\sqrt{3})(\sqrt{2}+\sqrt{3})$.
11. $(2^{\frac{1}{3}}+3^{\frac{1}{3}})^3$.
12. $\left(12-(19)^{\frac{1}{2}}\right)^{\frac{1}{3}}\left(12+(19)^{\frac{1}{2}}\right)^{\frac{1}{3}}$.
13. $\left(\frac{3-\sqrt{13}}{2}\right)^2-3\left(\frac{3-\sqrt{13}}{2}\right)-1$.
14. $(4x^2-2x\cdot 2^{\frac{1}{2}}+1)(4x^2+2x\cdot 2^{\frac{1}{2}}+1)$.

15. $(a+b)^{\frac{2}{3}} \cdot (a+b)^{\frac{1}{4}}(a+b)^{-\frac{1}{3}}$.

16. $a^{\frac{1}{n}}(a^2)^{\frac{1}{n}}(a^3)^{\frac{1}{n}}$.

17. $\dfrac{9\sqrt{2}}{4\sqrt{6}}$.

18. $\dfrac{\sqrt[3]{3}}{\sqrt{2}}$.

19. $\dfrac{\sqrt{2} \cdot \sqrt[3]{3}}{\sqrt[6]{6}}$.

20. $\dfrac{4 \cdot 9^{\frac{1}{3}}}{2 \cdot 3^{\frac{1}{2}}}$.

21. $\left(\dfrac{x}{y}\right)^{\frac{1}{2}} \div \left(\dfrac{y}{x}\right)^{\frac{1}{2}}$.

22. $\dfrac{3}{2^{\frac{1}{2}}} \div \dfrac{2^{\frac{1}{2}}}{3}$.

23. $a \div \dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$.

24. $\dfrac{(ax)^{\frac{1}{2}}}{(bx)^{\frac{1}{4}}}$.

25. $\dfrac{5^{\frac{2}{3}}}{5^{\frac{1}{3}}}$.

26. $\dfrac{5^{\frac{1}{2}}}{5^{\frac{1}{3}}}$.

27. $\dfrac{a^{\frac{1}{n}} \cdot a^{\frac{2}{n}}}{a^{\frac{3}{n}}}$.

28. $\dfrac{a^n \cdot a^{\frac{n}{2}}}{a^{\frac{n}{3}}}$.

Reduce to the simplest form.

29. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt[4]{4}}$.

30. $\dfrac{7-\sqrt[4]{9}}{7+\sqrt[4]{9}}$.

31. $\left(\dfrac{1}{2}\right)^{\frac{1}{2}}$.

32. $(32)^{\frac{1}{10}}$.

33. $\dfrac{7-3^{\frac{1}{2}}}{7+3^{\frac{1}{2}}}$.

34. $\dfrac{3 \cdot 7^{\frac{1}{2}} + 3^{\frac{1}{2}}}{2 \cdot 7^{\frac{1}{2}} - 3^{\frac{1}{2}}}$.

35. $\dfrac{a^{\frac{3}{2}} - a^{-\frac{3}{2}}}{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}$.

Evaluation of Formulas Involving Exponents and Radicals

36. The diagonal of a square of side, s, is $s\sqrt{2}$. Find the diagonal of a square of side 100 to three significant figures.

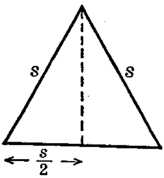

FIG. 11

37. The diagonal of a square of area, A, is given by $\sqrt{2A}$. What is the diagonal of a square of area 70 square inches?

38. The volume, V, of a cube is given by the formula $V = d^3 2^{-\frac{3}{2}}$, where d is the diagonal of a face. Compute V when $d = 120$ inches.

39. Find the area of an equilateral triangle whose side is 24. (Fig. 11.)

40. Find the side s of an equilateral triangle whose area is 36.

41. The probable error of the arithmetic mean of n measurements is given by

$$\frac{0.6745\sigma}{n^{\frac{1}{2}}}$$

where σ is called the standard deviation of the measurements. When $n = 1000$ and $\sigma = 1.850$, find the probable error to two significant figures.

42. The sag of an overhead trolley wire in an electric tramway is given by the formula

$$d = \sqrt{\frac{3L(l-L)}{8}},$$

where d is the number of feet in the sag, l is the number of feet of wire between poles, L is the number of feet from pole to pole. Find the sag when poles are 97 feet apart and length of wire is 97.2 feet.

43. The time in seconds required for the discharge of water from one vessel to another through an orifice in the side is

$$t = \frac{0.116 \cdot A \cdot B \cdot (F^{\frac{1}{2}} - f^{\frac{1}{2}})}{(A+B) \cdot a},$$

where F and f are the differences in the heights of water in the two vessels at the beginning and end respectively of the discharge, a is the area of the orifice, A is the area of a horizontal section of the discharging vessel, and B is that of the receiving vessel (measurements in inches).

Find t when $F = 188$, $f = 99$, $A = 94$, $B = 68$, and the orifice is a circle one and one half inches in diameter.

44. The probable error in a correlation coefficient, r, computed from n pairs of values is given by

$$\frac{0.6745(1 - r^2)}{n^{\frac{1}{2}}}.$$

Compute this probable error to two significant figures when $n = 1000$, $r = 0.675$.

45. The probable error in the coefficient of variability, C, of n measurements is given by

$$\frac{0.6745 \cdot C \cdot (1 + 2C^2)^{\frac{1}{2}}}{(2n)^{\frac{1}{2}}}.$$

Compute this probable error when $C = 0.3500$ and $n = 1000$.

46. The area in square feet of the top of a well-designed chimney is given by the formula

$$A = 0.03\frac{Q}{\sqrt{h}},$$

where Q is the quantity of coal in pounds used per hour and h is the height of the chimney. What should be the area of the top of a chimney 170 feet high which is connected with a furnace using 12,500 pounds of coal per hour?

47. The quantity of water in cubic feet per second flowing through a rectangular weir is given by the formula

$$Q = 3.33 \cdot [L - 2h]h^{\frac{3}{2}},$$

where h is the depth of water over the sill of the weir in feet, and L the length of the sill.

Find Q, where $L = 47$, $h = 1.7$.

48. Three equal uniform rods of weight w and of length l are jointed together to form a triangle ABC; this triangle is hung up by the joint A, and a weight W is attached to B and C by two strings of length $\frac{l}{\sqrt{2}}$. The compression in BC is given by

$$x = W\frac{\sqrt{3} + 1}{2\sqrt{3}} + \frac{w}{\sqrt{3}}.$$

Obtain x correct to nearest unit when $W = 300$, $w = 100$.

49. The area of a triangle whose sides are a, b, c is given by the formula

$$\sqrt{s(s - a)(s - b)(s - c)}$$

where $s = \frac{a + b + c}{2}$. Calculate from this formula the area of a triangle whose sides are 7, 13, and 14 inches.

CHAPTER VII

QUADRATIC EQUATIONS

48. Typical form. Any equation of the second degree (Art. 25) in one unknown x can, by transforming and collecting terms, be written in the typical form

$$ax^2 + bx + c = 0,$$

where a, b, c do not involve x, and have any values with the one exception that a is not zero. Since the result of multiplying the members of an equation in this typical form by any given number is an equation in typical form, the a, b, c can be selected in an indefinitely large number of ways. For example, a may be chosen to be an integer.

The function $ax^2 + bx + c$ $(a \neq 0)$ is called the typical **quadratic function.**

EXERCISES

Arrange the following equations in the typical form and select a, b, and c from the resulting equations. Write each equation so that a is an integer.

1. $3x^2 - x + k = \frac{2}{3}x^2 + 2.$

Solution: Transposing and collecting terms,

$$\frac{7}{3}x^2 - x + k - 2 = 0.$$

Hence $$a = \frac{7}{3}, \quad b = -1, \quad c = k - 2.$$

One way of writing the equation with a as an integer is

$$7x^2 - 3x + 3k - 6 = 0.$$

Another form is $$x^2 - \frac{3}{7}x + \frac{3}{7}k - \frac{6}{7} = 0.$$

2. $x^2 + (2x - 5)^2 = 3x.$

3. $3x^2 + 4 + \frac{x}{2} = \frac{x^2}{3} + x + 1.$

4. $\frac{x^2}{2} + \frac{x}{3} + \frac{1}{4} = 2x^2 + 3x - 4.$

5. $x^2 - 3kx + m = 2x - m.$

6. $(2x + m)^2 = 3(2x + m).$

7. $(x - 1)(x - 2) + (x - 1)(x - 3) + (x - 2)(x - 3) = 0.$

8. $\frac{(x + 1)^2}{4} + \frac{(x - 2)^2}{9} = r^2.$

49. Solution by factoring. When the left-hand member of a quadratic equation in typical form (Art. 48) can be factored readily, the solutions are easily obtained. Take, for example, the equation $x^2 - 4x = 21$ which in typical form is

$$x^2 - 4x - 21 = 0.$$

The factors of the left-hand member are easily found to be $(x + 3)$ and $(x - 7)$, and we may write the equation in the form

$$(x + 3)(x - 7) = 0.$$

Any value of x which makes either factor zero will satisfy the equation. If $x = -3$, we have

$$(-3 + 3)(-3 - 7) = 0 \cdot (-10) = 0.$$

Again if $x = 7$, we have

$$(7 + 3)(7 - 7) = 10 \cdot 0 = 0.$$

Hence -3 and 7 are the solutions of the given quadratic equation.

EXERCISES

Solve the following equations by factoring.

1. $(x - 3)^2 = 6 - 2x$.

Solution: Arranged in typical form this equation becomes

$$x^2 - 4x + 3 = 0.$$

The factors of the left-hand member are $(x - 3)$ and $(x - 1)$ and the equation may be written

$$(x - 3)(x - 1) = 0.$$

The solutions are 3 and 1.

2. $x^2 + 6x + 5 = 0$.

3. $(x + 3)^2 = 1$.

4. $(x + 1)(x - 1) - 8 = 0$.

5. $2x^2 + 5x - 3 = 0$.

6. $3x^2 + 4x + 1 = 0$.

7. $x^2 - nx = mn - mx$.

8. $3t^2 + 8t = 3$.

9. $s^2 - 2ns + n^2 = 0$.

10. $4n^2 - 9 = 0$.

11. $3z^2 - 7z = 0$.

12. $4 + x(4x - 17) = 0$.

50. Solution by formula. A quadratic equation may be solved by the process of "completing the square."

For example, to solve

$$3x^2 + 5x - 2 = 0,$$

write the equation in the form

$$x^2 + \tfrac{5}{3}x = \tfrac{2}{3}.$$

Add $(\frac{1}{2} \cdot \frac{5}{3})^2 = \frac{25}{36}$ to both members, and the left-hand member is a perfect square. We have then

$$x^2 + \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{2}{3} + \tfrac{25}{36} = \tfrac{49}{36},$$

or

$$(x + \tfrac{5}{6})^2 = \tfrac{49}{36}.$$

Extracting the square root of both sides,

$$x + \tfrac{5}{6} = \pm \tfrac{7}{6},$$

$$x = -2 \text{ or } \tfrac{1}{3}.$$

Both of these values of x satisfy the original equation. Thus

$$3(-2)^2 + 5(-2) - 2 = 3 \cdot 4 - 10 - 2 = 0,$$

$$3(\tfrac{1}{3})^2 + 5(\tfrac{1}{3}) - 2 = 3 \cdot \tfrac{1}{9} + \tfrac{5}{3} - 2 = 0.$$

Apply this method to the general quadratic equation

$$ax^2 + bx + c = 0.$$

Transpose c and divide through by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ to both members to make the left-hand member a perfect square,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extract the square root, and obtain

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, the roots of the general quadratic equation

$$ax^2 + bx + c = 0$$

are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If we denote the first of these roots by r_1 and the second by r_2, we may conveniently use these expressions as formulas for the solution of any quadratic. Thus, to solve the equation

$$3x^2 + 5x - 2 = 0,$$

we substitute in the formula, $a = 3$, $b = 5$, $c = -2$ and find

$$r_1 = \frac{-5 + \sqrt{25 - 4 \cdot 3 \cdot (-2)}}{6} = \frac{-5 + \sqrt{49}}{6} = \frac{1}{3}.$$

Similarly

$$r_2 = \frac{-5 - \sqrt{49}}{6} = -2.$$

EXERCISES

Solve the following equations by use of the formula, and verify by substitution.

1. $x^2 - 6x - 7 = 0.$
2. $2x^2 + 3x + 1 = -2x + 4.$
3. $5x^2 - 3x - 2 = 0.$
4. $s^2 + 2s = 120.$
5. $x^2 + 22x = -120.$
6. $2x(x + 4) = 42.$
7. $6n^2 - 5n - 6 = 0.$
8. $2m^2 + 3m = 27.$
9. $18t^2 + 6t = 4.$
10. $s(s + 4) = 7.$
11. $0.2x^2 + 0.9x = 3.5.$
12. $0.3x^2 - 0.7x = 1.$
13. $8s - 10 = s^2.$
14. $\dfrac{2x}{x+2} + \dfrac{x+2}{2x} = 2.$
15. $\dfrac{1}{2}x - \dfrac{3}{4}x^2 + 2 = 0.$
16. $x + 11 + \dfrac{1}{x} = \dfrac{27}{2}.$
17. $(x + 1)^2 - 8(x + 1) = 16.$
18. $3r^2 + r = 200.$
19. $\dfrac{2x - 2}{5x + 5} = \dfrac{x - 1}{x + 1}.$
20. $x^2 + (5 - x)^2 = (5 - 2x)^2.$
21. $(1 - e^2)x^2 - 2mx + m^2 = 0.$ Solve first for x in terms of m and e, then for m in terms of x and e, and finally for e in terms of x and m.
22. $\begin{vmatrix} x & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & x \end{vmatrix} = 0.$
23. $\begin{vmatrix} x & a & b \\ a & b & a \\ b & a & x \end{vmatrix} = 0.$
24. $x^2 - \sqrt{2}x + \sqrt{3}x = \sqrt{6}.$
25. $x^2 - 2ax = 2ab - bx.$
26. $x^2 - ax + a = x + ax - a^2.$
27. $x^2 - bx - 2x = ax - ab - 2a.$
28. $x^2 - 2nx = 1 + n - n^2.$
29. $6x^2 + 2mx - 3nx = mn.$
30. Show by substitution that

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are roots of $ax^2 + bx + c = 0.$

Solve the following equations to two significant figures.

31. $x^2 + 1.81x - 7.11 = 0.$

32. $x^2 + 8.90x + 12.3 = 0.$

33. $x^2 - 9.02x + 19.7 = 0.$

34. $0.911x^2 + 1.01x - 33.3 = 0.$

51. Equations in the quadratic form. If in an equation we can replace an expression containing the unknown by a new letter and have a quadratic equation in that letter, then the original equation is said to be in the **quadratic form.** Thus in the equation

$$(x^2 - 1)^2 - 11(x^2 - 1) + 24 = 0$$

if we let $z = x^2 - 1$, we obtain $z^2 - 11z + 24 = 0$. Again, if we let $u = x^{-\frac{3}{2}}$ in the equation

$$2x^{-3} + x^{-\frac{3}{2}} + 1 = 0, \text{ we have}$$
$$2u^2 + u + 1 = 0.$$

EXERCISES

Solve the following equations and check the results.

1. $x - 3 - \sqrt{x - 3} - 2 = 0.$

Solution: Let $u = \sqrt{x - 3}$, where the radical stands for the positive square root of the number under it. The equation then becomes

$$u^2 - u - 2 = 0,$$

or

$$u = 2 \quad \text{or} \quad -1.$$

Replacing u by its value in terms of x, we have

$$\sqrt{x - 3} = 2,$$
$$\sqrt{x - 3} = -1.$$

Since $\sqrt{x - 3}$ is the positive square root of $(x - 3)$, the equation $\sqrt{x - 3} = -1$ must be discarded. From $\sqrt{x - 3} = 2$ we have $x - 3 = 4$, or $x = 7$.

Check: $7 - 3 - \sqrt{7 - 3} - 2 = 4 - \sqrt{4} - 2 = 4 - 2 - 2 = 0.$

Hence, the result $x = 7$ satisfies the equation.

2. $x^4 - 3x^2 - 4 = 0.$

3. $x^4 - 10x^2 + 9 = 0.$

4. $x^6 - 9x^3 + 8 = 0.$

5. $(x^2 - 1)^2 - 11(x^2 - 1) + 24 = 0.$

6. $x + \sqrt{x} = 12.$

7. $\dfrac{1}{x^2} - \dfrac{1}{x} = 12.$

8. $6\sqrt{x} + \dfrac{1}{\sqrt{x}} = 5.$

9. $x - \sqrt{x - 3} - 23 = 0.$

Hint: Write the equation in exercise 9 in the form $x - 3 - \sqrt{x - 3} - 20 = 0.$

10. $2x + \sqrt{2x + 1} = 11.$

11. $3\sqrt{\dfrac{x - 1}{x + 1}} - 10\sqrt{\dfrac{x + 1}{x - 1}} + 29 = 0.$

12. $x^2 + x + 16 - 8\sqrt{x^2 + x + 4} = 0.$
13. $(x^2 + x)^2 - 2(x^2 + x) - 3 = 0.$
14. $x^3 - 9x^{\frac{3}{2}} + 8 = 0.$
15. $x^{-2} - 9x^{-1} + 20 = 0.$
16. $ax^4 + bx^2 + c = 0.$
17. $4\left(1 + \frac{1}{x}\right)^2 - 4\left(1 + \frac{1}{x}\right) = 3.$
18. $x^2 + 10 - 6\sqrt{x^2 + 1} = 0.$
19. $x^4 - 8x^3 + 23x^2 - 28x - 8 = 0.$

Hint: Write the equation in the form

$$x^4 - 8x^3 + 16x^2 + 7(x^2 - 4x) - 8 = 0.$$

20. $x^4 - 4x^3 + 2x^2 + 4x - 3 = 0.$
21. $x^4 + 2x^3 + x^2 - 4 = 0.$
22. $3x + 5 - \sqrt{1 - 3x} = 0.$
23. $\dfrac{5x}{x^2 + 1} + \dfrac{10x^2 + 10}{x} = 27.$
24. $a(ax + b)^2 + b(ax + b) + c = 0.$
25. $\sqrt{x + 10} + \sqrt[4]{x + 10} = 2.$

52. Theorems concerning the roots of quadratic equations.

THEOREM I. *If r is a root of the equation*

$$ax^2 + bx + c = 0, \tag{1}$$

then $(x - r)$ is a factor of $ax^2 + bx + c$. Conversely, if $(x - r)$ is a factor of $ax^2 + bx + c$, then r is a root of the equation.

If r is a root of the equation, then

$$ar^2 + br + c = 0. \quad \text{(Why?)} \tag{2}$$

We may now write

$$\begin{aligned} ax^2 + bx + c &= ax^2 + bx + c - (ar^2 + br + c) \quad \text{(Why?)} & (3)\\ &= a(x^2 - r^2) + b(x - r) & (4)\\ &= (x - r)(ax + ar + b) & (5)\end{aligned}$$

Hence, $(x - r)$ is a factor of $ax^2 + bx + c$.

Conversely, if $(x - r)$ is a factor of $ax^2 + bx + c$, then the substitution of r for x makes the factor $(x - r)$ vanish. Hence $ax^2 + bx + c$ takes on the value zero and r is a root of

$$ax^2 + bx + c = 0.$$

The student should study the special case for $r = 0$.

EXERCISES

Form quadratic equations of which the following are roots.

1. 3, 1.

Solution: When the right-hand member of the equation to be formed is 0, the left-hand member has factors $(x - 3)$ and $(x - 1)$. Hence,

$$(x - 3)(x - 1) = x^2 - 4x + 3 = 0$$

is a quadratic equation with roots 1 and 3. There are, of course, an indefinite number of other quadratic equations having 1 and 3 for roots, for we can multiply through by any number; for example, $2x^2 - 8x + 6 = 0$, $3x^2 - 12x + 9 = 0$, have roots 1 and 3.

2. 3, 2.

3. -1, 3.

4. 1, -3.

5. -1, -3.

6. 7, 0.

7. $-\frac{1}{2}, \frac{1}{3}$.

8. $\sqrt{2}$, 2.

9. $2 + \sqrt{3}$, $2 - \sqrt{3}$.

10. $1 + i$, $1 - i$, where $i^2 = -1$. (See Art. 47.)

11. $1 - 2i$, $1 + 2i$.

12. $5, \frac{1}{5}$.

13. $\frac{a}{b}, \frac{b}{a}$.

14. $a - b$, $a + b$.

15. Verify by performing the indicated operations that

$$a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = ax^2 + bx + c.$$

53. Number of roots. In order to avoid certain exceptions, an equation $f(x) = 0$ is said to have as many roots as $f(x)$ has factors of the type $(x - r_1)$ where r_1 is any number. A factor $(x - r_1)$ may be repeated. For example, if $(x - r_1)^2$ is a factor of $f(x)$, we say that $f(x) = 0$ has two roots equal to r_1.

We have shown that a quadratic equation has two roots. The question arises: has it *only* two or may it have more? This question is answered by the following

THEOREM II. *A quadratic equation has only two roots.*

Proof. Suppose there is, in addition to

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

a third root r_3, distinct from r_1 and r_2, of the equation

$$ax^2 + bx + c = 0.$$

By exercise 15, Art. 52, $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

Hence if r_3 is a root,

$$a(r_3 - r_1)(r_3 - r_2) = 0.$$

But this is impossible since no one of these factors is zero. (III, Art. 5.)

54. Special forms of quadratics. In the typical quadratic $ax^2 + bx + c = 0$, c is often called the **known term,** and bx the **term in x**. Either the known term or the term in x or both may not be present, but we still have a quadratic equation though it consists of only one or two terms. Such quadratics are often called **incomplete** quadratics.

If $c = 0$, $ax^2 + bx + c = 0$ becomes $ax^2 + bx = 0$. Since x is a factor of $ax^2 + bx$, we have one root equal to 0. If both b and c are 0, the equation becomes $ax^2 = 0$. Now x^2 is a factor, or x is a factor twice, and we have two roots equal to 0.

If $b = 0$, but $c \neq 0$, $ax^2 + bx + c = 0$ reduces to $ax^2 + c = 0$. In this case, $x = \pm\sqrt{-\frac{c}{a}}$. That is, the roots are arithmetically equal, but opposite in sign.

EXERCISES

Determine k so that each of the following equations shall have one root equal to zero.

1. $3x^2 + 6x - 5 + 2k = 0$.

Solution: One root only of the equation $ax^2 + bx + c = 0$ is zero when $c = 0$ and a, b are different from zero. In this exercise, $a = 3$, $b = 6$, $c = -5 + 2k$. In order for c to be zero, k must equal $\frac{5}{2}$.

2. $2x^2 - \frac{x}{2} + k - 4k^2 = 0$.

3. $x^2 + 9x + k^2 - 2k - 3 = 0$.

Determine k and m so that each of the following equations shall have two roots equal to zero.

4. $3x^2 + 8mx + 2kx + m - k + 1 = 0$.

5. $y^2 + 3my - ky + y + k - 1 = 0$.

Determine k so that the roots of the following equations may be arithmetically equal but opposite in sign.

6. $4x^2 + 2kx + x + 5 = 0$.

7. $9x^2 + k^2x + kx - 2x = 1$.

55. Nature of the roots. In Art. 50, we found the two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

to be $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

In case a, b, c are real numbers, the numerical character of these roots depends upon the number $b^2 - 4ac$ under the radical sign. An examination of r_1 and r_2 leads at once to the following conclusions:

(1) If $b^2 - 4ac > 0$, the roots are real and unequal.
(2) If $b^2 - 4ac < 0$, the roots are imaginary and unequal.
(3) If $b^2 - 4ac = 0$, the roots are real and equal.

It should be observed that if the coefficients are real and one root is imaginary, then both roots are imaginary.

The quantity $b^2 - 4ac$ is called the **discriminant** of the equation

$$ax^2 + bx + c = 0.$$

56. Sum and product of the roots. If we add together the two roots of $ax^2 + bx + c = 0$, we have

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

If we multiply the two roots together, we have

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}.$$

Hence:

I. *The sum of the roots of a quadratic equation in x is equal to the coefficient of x with its sign changed, divided by the coefficient of x^2.*

II. *The product of the roots of a quadratic equation in x is equal to the known term divided by the coefficient of x^2.*

EXERCISES

Determine the nature of the roots of the following equations.

1. $3x^2 + 11x - 4 = 0$.

2. $x^2 + 2x + 3 = 0$.

3. $25x^2 - 20x + 4 = 0$.

4. $6x^2 - 4x + 3 = 0$.

5. $7x^2 + 4 = 0$.

6. $a^2x^2 + b^2x = c^2$.

Determine the real values of k so that the two roots of each of the following equations may be equal.

7. $8x^2 + 8kx + 3k + 2 = 0$.

Solution: In order that the roots of this equation may be equal, it is necessary that $b^2 - 4ac = 64k^2 - 96k - 64 = 0$. Hence, k must be a solution of $64k^2 - 96k - 64 = 0$, or $2k^2 - 3k - 2 = 0$, or $k = -\frac{1}{2}$, or 2. Substituting $k = -\frac{1}{2}$ in the above equation, we get

$$8x^2 - 4x + \frac{1}{2} = \frac{1}{2}(4x - 1)^2 = 0.$$

With $k = 2$, we get

$$8x^2 + 16x + 8 = 8(x + 1)^2 = 0.$$

8. $x^2 + 3kx + k + 7 = 0.$
9. $x^2 + 2kx + 3 = 0.$
10. $2x^2 + 6x + k = 0.$
11. $kx^2 + 2x + 3 = 0.$
12. $kx^2 + (k + 1)x + k = 0.$
13. $kx^2 + kx + 1 = 0.$
14. $kx^2 + x + k = 0.$
15. $x^2 + kx + k = 0.$
16. $8x = (2x + k)^2.$
17. $(k - 1)x^2 + kx + k + 1 = 0.$

Determine by inspection the sum and product of the roots of the following equations.

18. $3x^2 - 2x + 7 = 0.$
19. $5x^2 - 8x + 10 = 0.$
20. $x^2 + 13x - 5 = 0.$
21. $7 - 3x - 2x^2 = 0.$
22. $s^2x^2 - a^2x + r^2 = 0.$
23. $\frac{1}{2}gt^2 + v_0t = d.$
24. $\frac{x^2}{a} + \frac{x}{b} + \frac{1}{c} = 0.$

Determine the value of k in the following equations.

25. $3x^2 - 20x + k = 0$, where one root is 7.

Hint: See Art. 20 where "root of an equation" is defined. Can this exercise be solved by using the theorems of Art. 56?

26. $x^2 + kx + 7 = 0$, where one root is 1.

27. $4x^2 - 16x + 3k = 0$, where the difference between the roots is 5.

28. $7x^2 + kx - 12 = 0$, where a quotient of the two roots is $-\frac{3}{7}$.

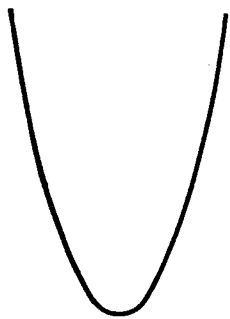

FIG. 12

57. Graph of the quadratic function. In Chapter III we have plotted certain quadratic functions. It can be shown, if a is positive and different from zero, that the graph of the function $ax^2 + bx + c$ has the same general characteristics as the curve in Fig. 12. This curve is called a parabola. The real roots of the equation $ax^2 + bx + c = 0$ are given by the abscissas of the points where the curve crosses the X-axis. If the curve has no point in common with the axis, then the roots of the equation are imaginary. For we have shown that every quadratic equation has two roots, real or imaginary. If the curve touches the X-axis, both roots of the equation are real and equal. These three cases are shown in Fig. 13, where the graphs of $x^2 - 2x - 3$, $x^2 - 2x + 1$, and $x^2 - 2x + 5$ are given.

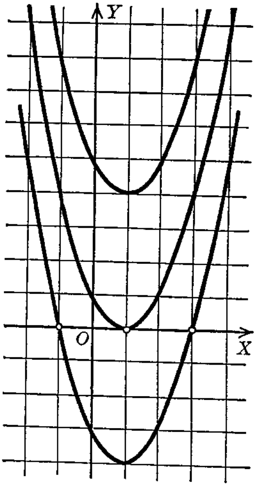

FIG. 13

EXERCISES

Construct the graphs of the functions in the following equations, and, by measurement, determine the roots if they are real. Calculate the value of the function for at least ten values of x between the limits given. Choose the vertical unit of such a length that the graph will be of convenient proportions for the coördinate paper.

1. $x^2 - 4x + 3 = 0$, from $x = 0$ to $x = 4$.
2. $x^2 + x - 12 = 0$, from $x = -5$ to $x = 4$.
3. $4x^2 + 12x + 5 = 0$, from $x = -4$ to $x = 1$.
4. $x^2 - 4x = 0$, from $x = -1$ to $x = 5$.
5. $x^2 + 2x + 2 = 0$, from $x = -3$ to $x = 2$.
6. $x^2 - 6x - 7 = 0$, from $x = -2$ to $x = 8$.
7. $6 - 3x - x^2 = 0$, from $x = -5$ to $x = 2$.
8. $x^2 + 2x - 4 = 0$, from $x = -4$ to $x = +2$.
9. $4 - 5x - x^2 = 0$, from $x = -7$ to $x = 2$.
10. $x^2 - 4 = 0$, from $x = -3$ to $x = +3$.
11. What are the general characteristics of the graph of the function $ax^2 + bx + c$ if a is negative? If $b = 0$? If $c = 0$?

PROBLEMS

1. Find two consecutive positive integers whose product is 462.
2. Find two consecutive positive even integers whose product is 1368.
3. In the course of Steinmetz's solution of the problem of finding the current strength in a divided electric circuit, it is necessary to solve the equation

$$a^2x^2 - as^2 + r^2 = 0$$

for a. His solution is

$$\frac{s^2 \pm q^2}{2x^2},$$

where $q^2 = \sqrt{s^4 - 4r^2x^2}$. Verify the result.

4. If a ball is thrown upward with a velocity v_0, the distance d from the earth to the ball after a given time t is given by the formula

$$d = v_0 t - \frac{1}{2} g t^2, \tag{1}$$

where $g = 32.2$. The speed at the time t is given by

$$v_t = v_0 - gt. \tag{2}$$

If the ball is thrown *downward* with a speed v_0, the above formulas become

$$d = v_0 t + \frac{1}{2} g t^2, \tag{3}$$

$$v_t = v_0 + gt. \tag{4}$$

If a ball is thrown upward with a velocity 60 feet per second, in what time will it be just 40 feet from the ground? Explain the two answers.

5. How long will it take the ball described in problem 4 to reach the ground?

Hint: Put $d = 0$ in formula (1).

6. At what time is the velocity of the ball zero?

7. How high will the ball rise?

8. How far does the ball rise in the second second?

9. How long will it take a ball to fall 500 feet, if it is thrown downward with an initial speed of 50 feet per second?

10. How much longer would it take the ball in problem 9 to fall 500 feet if it had been dropped with no initial velocity?

11. If a body falls from rest, how far will it fall during the fifth second?

12. If a body is thrown downward with an initial velocity of 10 feet per second, how far will it fall during the fifth second?

13. A drives his car 5 miles per hour faster than B and covers 180 miles in one half hour less than B. Find the speed of A's car.

14. By increasing the radius of a sphere 1 inch, its volume is increased 10 cubic inches. Find the radius of the original sphere to two decimal places. (Volume of a sphere $= \frac{4}{3}\pi r^3$.)

15. The edges of a cube are each increased in length 1 inch. It is found that the volume is thereby increased 10 cubic inches. What was the length of the edge of the cube?

16. The diagonal of a square is one unit longer than the side of the square. What is the length of a side?

17. A rectangular sheet of tin whose dimensions are a and b has square corners cut out, and the sides turned up to form a box. The box will have a maximum volume if the depth x is a root of $12x^2 - 4(a + b)x + ab = 0$. Find this depth. Also find the depth when the rectangle is a square of side a.

18. Show that the equation

$$x^2 + bx + c = 0$$

has one positive and one negative root if b is real and c negative.

19. In joining together two steel boiler plates with a single row of rivets, the distance p between the centers of the rivets is given by the formula

$$p = 0.56\frac{d^2}{t} + d,$$

where t is the thickness of the plate and d the diameter of the rivet holes. In a boiler the rivets are to be placed $1\frac{1}{2}$ inches apart. If the thickness of the plate is $\frac{3}{16}$ inch, what is the diameter of the rivet holes?

20. Graph on the same set of coördinate axes the function $2x^2 - x + c$, where c takes on the values $2, -4, 0, 8$. What effect does changing the constant term in a quadratic function have on the graph?

21. Graph on the same sheet the function $ax^2 - x + 3$, where a takes the values $5, 1, \frac{1}{2}, \frac{1}{10}, \frac{1}{100}$. Decreasing the coefficient of x^2 toward zero has what effect on the graph? What is the effect on the roots of the quadratic equation $ax^2 - x + 3 = 0$, if a is made to approach 0?

22. If s is the area in square inches of the flat end of a boiler, and t the thickness of the boiler plate in sixteenths of an inch, then the pressure p per square inch which the flat end plate can safely sustain is given by the formula

$$p = \frac{200(t + 1)^2}{s - 6}.$$

What should be the thickness to the nearest sixteenth of an inch of the boiler plate for the end of a boiler 20 inches in diameter to sustain a pressure of 100 pounds per square inch?

23. Let h be the height and t the thickness (in feet) of a rectangular masonry retaining wall. For very sandy soil with a grade angle of 20°, h and t are connected by the equation

$$t^2 + 0.19t \cdot h - 0.18h^2 = 0.$$

What should be the thickness (to the nearest inch) of a retaining wall four feet high?

24. For loam, the equation in problem 23 would be

$$t^2 + 0.14t \cdot h - 0.13h^2 = 0.$$

What should be the thickness of a retaining wall four feet high?

25. A long horizontal pipe is connected with the bottom of a reservoir. If H be the depth of the water in the reservoir in feet, d the diameter of the pipe in inches, L the length of the pipe in feet, and v the velocity of the water in the pipe in feet per second, then according to Cox's formula

$$\frac{Hd}{L} = \frac{4v^2 + 5v - 2}{1200}.$$

Find the velocity of water in a 5-inch pipe, 1000 feet long, connected with a reservoir containing 49 feet of water.

26. In a group of points every point is connected with every other point by a straight line. There are 105 straight lines. How many points are there?

27. The so-called effective area of a chimney is given by

$$E = A - 0.6\sqrt{A},$$

where A is the measured area. Find A when E is 24 square feet.

28. The electrical resistance of a wire depends upon the temperature of the wire according to the formula

$$R_t = R_0(1 + at + bt^2),$$

where a and b are constants depending on the material, R_0 is the resistance at 0°, and R_t the resistance at t°. For copper wire $a = 0.00387$, $b = 0.00000597$, and $R_0 = 0.02057$. At what temperature is the resistance double that at 0°?

29. The radius of a cylinder is 10 and its height 4. How much can be added to either the radius or to the height, and yet give the same increase in volume?

The following equations occur in some electrical problems.

30. $g = \dfrac{R}{R^2 + X^2}\cdot$ Solve for R.

31. $T = \dfrac{a(n - n')}{1 + b(n - n')^2}\cdot$ Solve for $(n - n')$.

32. $P = \dfrac{RW(r^2 + x^2)}{r(Rr - Xx)}\cdot$ Solve for x.

33. In making war bread a mixture of rye and corn meal was used. From a hundred pounds of rye flour a certain amount was taken and replaced by corn meal. Later, from the mixture the same amount was removed and again replaced by corn meal. The resulting mixture was 16 parts rye to 9 parts corn. What were the proportions in the first mixture?

34. A stone is dropped into a well, and 4 seconds afterward the report of its striking the water is heard. If the velocity of sound is taken at 1190 feet per second, what is the depth of the well? (Use $g = 32.2$. See problem 4.)

35. A quadratic expression in x is positive except when $-1 \leqq x \leqq 3$. Another quadratic expression is always negative except when $-3 \leqq x \leqq 2$. When $x = 0$ both expressions take on the same numerical value but are opposite in sign. For what values of x are the two quadratic expressions equal?

CHAPTER VIII

SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

58. Quadratic equations in two unknowns. An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (1)$$

in which at least one of the numbers A, B, or C is not zero, is called the **general quadratic equation in x and y**, or the **general equation of the second degree in x and y.**

ORAL EXERCISES

By comparison with the general equation, (1), give the values of A, B, C, D, E, and F in each of the following:

1. $x^2 + y^2 - 25 = 0.$
2. $9x^2 + 16y^2 = 144.$
3. $xy = 25.$
4. $5x^2 - 12y^2 = 60.$
5. $xy - 4x - 6y - 16 = 0.$
6. $y = 3x^2 - 6x + 12.$

We shall be concerned in this chapter with systems of equations of certain special forms which will be found useful in solving some interesting problems.

59. One equation linear and one quadratic. Any system of two equations in two unknowns in which one equation is linear and the other is quadratic can be solved by elimination of one unknown. First solve the linear equation for one of the unknowns in terms of the other, and then substitute in the quadratic equation.

Example 1. Solve the system

$$x^2 + y^2 = 25, \qquad (1)$$

$$x - y = 1. \qquad (2)$$

Solution: Solving (2) for y in terms of x,

$$y = x - 1. \qquad (3)$$

Substituting $x - 1$ for y in (1),

$$x^2 + (x - 1)^2 = 25. \qquad (4)$$

From (4)

$$2x^2 - 2x - 24 = 0,$$

or

$$x^2 - x - 12 = 0. \qquad (5)$$

Solving (5) by the formula

$$x = \frac{1 \pm \sqrt{1 + 48}}{2}$$
$$= 4 \text{ or } -3.$$

Substituting 4 for x in (2), we obtain $y = 3$.

Substituting -3 for x in (2), we obtain $y = -4$.

This gives $\begin{cases} x = 4 \\ y = 3 \end{cases}$ and $\begin{cases} x = -3 \\ y = -4 \end{cases}$ for the solutions.

Check these solutions by substitution in (1) and (2).

Graphical meaning of the two solutions. The graph of

$$x - y = 1 \qquad (2)$$

is the straight line shown in Fig. 14, and the graph of

$$x^2 + y^2 = 25 \qquad (1)$$

is the circle there shown. To draw the graph of (1), the student may give various values to x and calculate the corresponding values for y from $y = \pm\sqrt{25 - x^2}$.

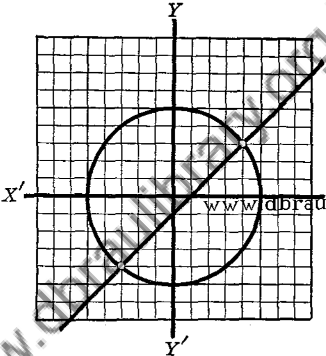

FIG. 14

Any point on the circle (1) has coördinates that satisfy equation (1). Any point on the straight line (2) has coördinates that satisfy equation (2). The points (4, 3) and (− 3, − 4) lie on both graphs, and satisfy both equations (1) and (2). That is to say, each point of intersection of the graph of (1) with the graph of (2) gives a pair of numbers that is a solution of the system.

Example 2. Solve the system

$$xy = 24, \qquad (1)$$
$$y - 2x + 2 = 0. \qquad (2)$$

Solution: Solving (2) for y in terms of x,

$$y = 2x - 2. \qquad (3)$$

Substituting $2x - 2$ for y in (1),

$$x(2x - 2) = 24, \qquad (4)$$
$$2x^2 - 2x - 24 = 0,$$
$$x^2 - x - 12 = 0,$$

or

$$(x + 3)(x - 4) = 0. \qquad (5)$$

Thus

$$x = 4 \text{ or } -3.$$

Substituting 4 for x in (1), we have

$$y = 6.$$

Substituting -3 for x in (1)

$$y = -8.$$

This gives $\begin{cases} x = 4 \\ y = 6 \end{cases}$ and $\begin{cases} x = -3 \\ y = -8 \end{cases}$ for solutions as may be verified by substitution in (1) and (2).

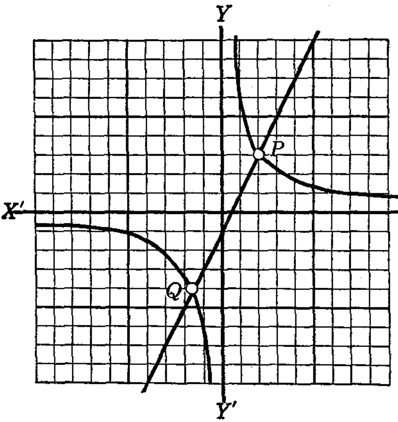

FIG. 15

Graphical meaning of the solutions. The graphs for equations (1) and (2) are shown in Fig. 15. The graph of

$$y - 2x + 2 = 0$$

is the straight line, and that of

$$xy = 24$$

is the curve with two branches as shown. This curve belongs to a class of curves called **hyperbolas.** The points of intersection, P and Q, have coördinates that are the solutions of the given system. All points on the graph of (1) have coördinates that satisfy equation (1). All points on the graph of (2) have coördinates that satisfy equation (2). Therefore, the points of intersection have coördinates that satisfy both equations.

Example 3. Solve the system

$$x^2 + y^2 = 25, \qquad (1)$$

$$x + y = 10, \qquad (2)$$

and draw the graph to explain the fact that the solutions are not real.

If the student carries out the same method as that illustrated in Example 1, he will obtain for solutions

$$x = 5 + \frac{5i}{2}\sqrt{2},$$

$$y = 5 - \frac{5i}{2}\sqrt{2};$$

and $$x = 5 - \frac{5i}{2}\sqrt{2},$$

$$y = 5 + \frac{5i}{2}\sqrt{2},$$

where $i^2 = -1$.

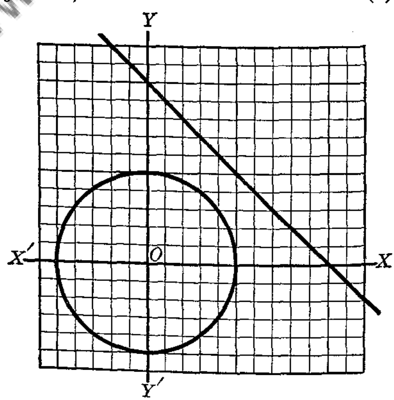

FIG. 16

Check these solutions by substitution in (1) and (2).

The graph of equation (1) is the circle shown in Fig. 16, and the graph of equation (2) is the straight line there shown. It is to be noted that these graphs do not intersect. This fact means that there exists no pair of real numbers that satisfies both equations (1) and (2).

60. Comments on the graphs of quadratic equations in two variables. In analytic geometry, a detailed study is made of the graphs of quadratic equations in two variables. A general notion of the shape of the graph of a quadratic equation in two variables may be given by considering the curve formed by the intersection of a plane and a right circular cone. On this account, the curves are called *conic sections* or merely *conics*. The cone used in this connection is the double cone as shown in Fig. 17. A conic may be a circle, an ellipse, a hyperbola, a parabola. In addition, the graphs of certain quadratic equations may be a pair of straight lines, or a point. The graphs of a few equations of standard forms will now be briefly discussed.

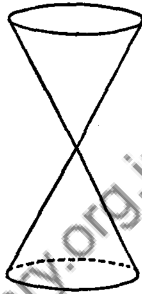
FIG. 17

(1) The graph of an equation of the form

$$Ax^2 + Ay^2 = C,$$

in which A and C have the same sign, is a circle with its center at the origin and with its radius equal to $\sqrt{\frac{C}{A}}$. Thus, $x^2 + y^2 = 9$ is a circle with its center at the origin and of radius 3.

(2) The graph of an equation of the form

$$Ax^2 + By^2 = C,$$

in which A, B, and C have the same signs, but $A \neq B$, is an oval-shaped figure called an **ellipse** with its center at the origin and with symmetry about the X- and Y-axes. Thus, the graph of $4x^2 + 9y^2 = 36$ shown in

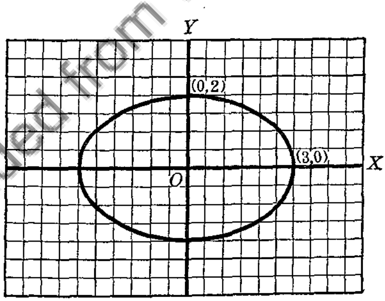

FIG. 18

Fig. 18 is an ellipse which crosses the X-axis at $(\pm 3, 0)$ and the Y-axis at $(0, \pm 2)$.

(3) The graph of an equation of the form

$$Ax^2 + By^2 = C,$$

in which A and B have opposite signs, and $C \neq 0$, represents a **hyperbola** with symmetry about the X- and Y-axes. The hyperbola has two separate parts (Fig. 19). Thus, the graph of $4x^2 - 9y^2 = 36$ shown in Fig. 19 is a hyperbola which crosses the X-axis at $(\pm 3, 0)$.

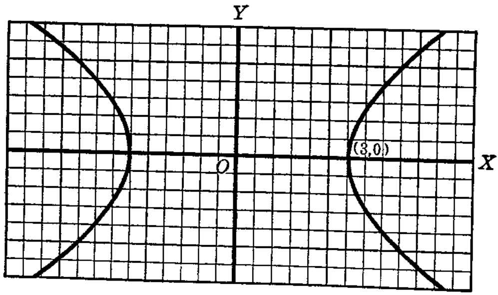

FIG. 19

(4) The graph of an equation of the form

$$y = ax^2 + bx + c$$

represents a **parabola** as explained in Art. 57.

EXERCISES AND PROBLEMS

Solve the following systems, verify each set of roots by substitution in the given equations, and draw the graphs in exercises 1, 2, 3, 7, 8, 9.

1. $x^2 + y^2 = 13,$
$x + y = 5.$

2. $x^2 + y^2 = 13,$
$2x + 3y = 13.$

3. $x^2 - y^2 = 48,$
$x - 7y = 0.$

4. $2x + 3y = 7,$
$xy + y^2 = 5.$

5. $3x - 2y = 1,$
$x^2 + y^2 + 6x - 2y = 6.$

6. $x^2 + 2y^2 = 8,$
$x + 2y = 6.$

7. $2x - y = 4,$
$4x^2 + 9y^2 = 72.$

8. $xy = 4,$
$y - 2x = 2.$

9. $3x^2 - 4y^2 = 12,$
$x - 2y = 2.$

10. $2x - 3y + 3 = 0,$
$4x^2 + 9y^2 - 6xy = 63.$

11. $4x + 3y = 3,$
$\frac{1}{x} + \frac{1}{y} = 5.$

12. $3y - 4x = 16,$
$\frac{6}{x} + \frac{4}{y} = 1.$

13. $3u + 4v = 12,$
$9u^2 + 16v^2 = 144.$

14. $0.3u + 1.125v = 3u,$
$2.25uv + 3.5v = 3u.$

Obtain to two significant figures the solutions of the following:

15. $xy = -0.24,$
$3.1x - 0.63y = 4.3.$

16. $y^2 - 4.1x = 0.38,$
$x - 0.37y = 0.29.$

17. The perimeter of a rectangular athletic field is 2248 yards, and the area is 64 acres. What are the dimensions?

18. The difference of the two legs of a right triangle is 7, the hypotenuse is 17. Find the sides of the triangle.

19. Find two consecutive integers the difference of whose squares is 31.

20. The difference of the areas of two squares is 900 square feet and the difference of their perimeters is 40 feet. Find a side of each square.

21. The sum of two numbers is 10, and their product is 9. Find the sum of their squares, and of their cubes.

Supplementary Exercises

Find the values of a, b, c, or r in the following exercises so that the straight line which is the locus of the first degree equation

(1) cuts the other locus in two distinct points,
(2) is tangent to the curve,
(3) fails to meet the curve.

22.

$$x^2 + y^2 = r^2,$$
$$3x + 4y = 5.$$

The locus of the first equation is a circle with center at the origin and a radius equal to r.

Solution: From the second equation, we have

$$x = \frac{5 - 4y}{3}.$$

Substituting in the first, we find

$$25y^2 - 40y + 25 - 9r^2 = 0.$$

Solving for y, we obtain, $y = \dfrac{40 \pm 30\sqrt{r^2 - 1}}{50}$.

If y is real, $r^2 - 1$ must be equal to or greater than zero.

Furthermore, if r is any number greater than 1, the two loci intersect in real and distinct points.

If $r = 1$, there is only one value for y, and the line is tangent to the circle. If $r < 1$, the line does not intersect the circle.

23. $x^2 + y^2 = r^2,$
$x + y = 2.$

24. $x^2 + y^2 = 25,$
$3x + 4y = c.$

25. $x^2 + y^2 = 25,$
$ax + 6y = 1.$

26. $x^2 + y^2 = 25,$
$7x - by = 3.$

27. For what values of b in terms of r and m does the system of equations

$$y = mx + b,$$
$$x^2 + y^2 = r^2$$

have equal solutions?

28. Determine the relation between a, b, m, and k such that the system

$$y = mx + k,$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has equal solutions.

29. Show that the straight line $3x + 4y = 25$ is tangent to the circle $x^2 + y^2 = 25$, and find the point of contact of this tangent.

61. Both equations quadratic. When both equations of a system are quadratic, the problem is often so difficult that the system cannot be solved by methods at our disposal.

As illustrated in the following example, the solution of a pair of quadratic equations often reduces to the solution of an equation of the fourth degree.

Example: Solve

$$x^2 + y^2 + x - 9 = 0,$$
$$x^2 + 2y^2 - 3y - 8 = 0.$$

Subtracting the second from the first, we have

$$-y^2 + 3y + x - 1 = 0,$$

or

$$x = 1 - 3y + y^2.$$

Substituting in the second equation, we have

$$(1 - 3y + y^2)^2 + 2y^2 - 3y - 8 = 0,$$

or

$$y^4 - 6y^3 + 13y^2 - 9y - 7 = 0.$$

At this stage of his progress in algebra, the student will not be able to solve a general equation of the fourth degree, hence he cannot proceed with the solution of this problem. There are, however, some forms of such equations for which we may easily obtain solutions. In Arts. 62 and 63 we shall consider a few such equations.

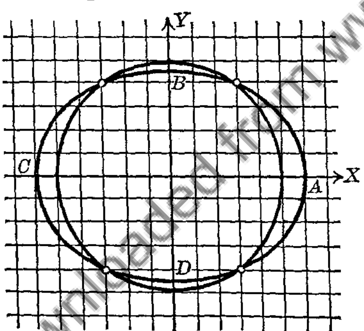

FIG. 20

When both equations are quadratic, the system ordinarily has **four different solutions.** When each of the four solutions is a pair of real numbers, the graphs intersect in four points whose coördinates are the solutions as in Fig. 20. When two of the four solutions are real numbers and two involve imaginary numbers, the graphs intersect in only two points. When each of the four solutions involves imaginary numbers, the graphs do not intersect.

62. Both equations of the form $ax^2 + by^2 + c = 0$. If, instead of considering x and y as the unknowns, we consider first x^2 and y^2 as the unknowns, the method of solution is that for linear equations.

EXERCISES

Solve the following systems of equations and give the graphical representations for exercises 1, 2, 3, 5, and 6:

1.
$$16x^2 + 27y^2 = 576,$$
$$x^2 + y^2 = 25.$$

Solution: Solving for x^2 and y^2 we have

$$x^2 = \frac{\begin{vmatrix} 576 & 27 \\ 25 & 1 \end{vmatrix}}{\begin{vmatrix} 16 & 27 \\ 1 & 1 \end{vmatrix}} = \frac{-99}{-11} = 9, \qquad y^2 = \frac{\begin{vmatrix} 16 & 576 \\ 1 & 25 \end{vmatrix}}{\begin{vmatrix} 16 & 27 \\ 1 & 1 \end{vmatrix}} = 16.$$

$$x = \pm 3. \qquad y = \pm 4.$$

Hence, we find the following four solutions,

$$(3, 4), (-3, 4), (3, -4), (-3, -4).$$

To show these solutions graphically, we plot the graphs of the two equations.

Solving each for y, we have $\quad y = \pm\sqrt{\dfrac{576 - 16x^2}{27}},$

and $\quad y = \pm\sqrt{25 - x^2}.$

The first equation has for its graph an ellipse, the second, the circle (Fig. 20). The points of intersection represent graphically the four pairs of solutions.

2. $x^2 + y^2 = 25,$
$9x^2 + 25y^2 = 225.$

3. $x^2 + 4y^2 = 25,$
$x^2 - 2y^2 = 1.$

4. $x^2 - y^2 = 15,$
$9x^2 + 16y^2 = 160.$

5. $x^2 + y^2 = 25,$
$9x^2 - 16y^2 = 0.$

6. $x^2 + y^2 = 9,$
$9x^2 + 16y^2 = 288.$

7. $4u^2 + 9v^2 = 144,$
$u^2 + v^2 = 25.$

Obtain to two significant figures the solutions of the following:

8. $x^2 + y^2 = 9.8,$
$4.3x^2 + 9.1y^2 = 88.$

9. $x^2 - y^2 = -4.1,$
$1.9x^2 + 0.21y^2 = 3.6.$

10. Find four pairs of numbers such that the sum and difference of their squares are respectively 265 and 23.

63. All terms containing unknowns are of the second degree. Equations in which each term containing an unknown is of the second degree have no first degree terms. Two methods of solution of such equations are illustrated by the following example.

Example: Solve
$$x^2 + y^2 = 20, \tag{1}$$
$$x^2 + 3xy = 28. \tag{2}$$

First solution: The main feature of the first solution is that we obtain a factorable second degree expression by the elimination of the known terms. To do this, we treat our example as follows:

Multiply (1) by 7: $7x^2 + 7y^2 = 140.$ (3)

Multiply (2) by 5: $5x^2 + 15xy = 140.$ (4)

Subtract, (3) − (4): $2x^2 - 15xy + 7y^2 = 0;$ or,

$$(y - 2x)(7y - x) = 0. \tag{5}$$

By equating each factor of (5) to zero, we obtain two linear equations:

$$y - 2x = 0 \quad \text{and} \quad 7y - x = 0.$$

It remains to solve each of these equations with (1) or (2), say with (1). Thus, we are to solve the two systems:

$$y - 2x = 0, \tag{6}$$
$$x^2 + y^2 = 20; \tag{7}$$

and

$$7y - x = 0, \tag{8}$$
$$x^2 + y^2 = 20. \tag{9}$$

Solving (6) and (7), we get the solutions (2, 4), (− 2, − 4), and solving (8) and (9), we get $\left(\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}\right)$, $\left(-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10}\right)$.

Second solution: The main feature of the second solution is the substitution of $y = mx$ in both equations.

By setting $y = mx$ in (1), we have

$$x^2 + m^2x^2 = 20,$$

whence

$$x^2 = \frac{20}{1 + m^2}. \tag{10}$$

By setting $y = mx$ in (2), we have

$$x^2 + 3mx^2 = 28,$$

whence

$$x^2 = \frac{28}{1 + 3m}. \tag{11}$$

Equating these values of x^2 in (10) and (11), we obtain

$$\frac{28}{1 + 3m} = \frac{20}{1 + m^2}.$$

Clearing of fractions and reducing, we obtain

$$7m^2 - 15m + 2 = 0,$$

or

$$m = 2, \text{ or } \frac{1}{7}.$$

Substituting these values of m in $x^2 = \dfrac{28}{1 + 3m}$,

we find, for $m = 2$, $\quad x^2 = 4,\ x = \pm 2,$

$$y = mx = \pm 4.$$

For $m = \frac{1}{7}$, we find $\quad x^2 = \frac{98}{5},\ x = \pm \frac{7}{5}\sqrt{10} = \pm 4.43^+,$

$$y = mx = \pm \frac{1}{5}\sqrt{10} = \pm 0.63^+.$$

The solutions are therefore

$(2, 4)$, $(-2, -4)$, $\left(\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}\right)$, $\left(-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10}\right)$.

The graphs of the two equations of this exercise are shown in Fig. 21. The geometrical interpretation of the substitution $y = mx$ is also shown in the figure.

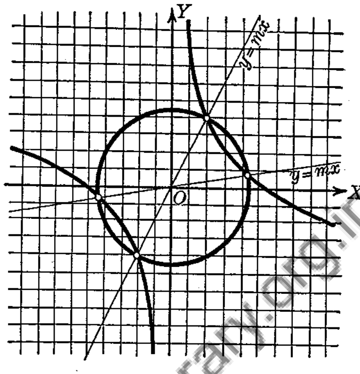

FIG. 21

EXERCISES

Solve each of the following systems of equations and plot the graph in the case of exercise 1:

1. $xy = 15,$ $x^2 - y^2 = 16.$

2. $x^2 + xy = 4,$ $y^2 + xy = 1.$

3. $xy + 4y^2 = 8,$ $x^2 + 3xy = 28.$

4. $2u^2 - 2uv = 15,$ $2v^2 + 6uv = -7.$

5. $2x^2 + 2y^2 - xy = 32,$ $x^2 + 3y^2 + 2xy = 19.$

6. $s^2 - 4t^2 = 9,$ $st + 2t^2 = 3.$

7. $x^2 - 3xy + 2y^2 = 8,$ $x^2 - xy - y^2 = 20.$

8. $x^2 + y^2 = 45,$ $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3}.$

9. $(x - y)(y + 2x) = 0,$ $9x^2 - 4y^2 = 36.$

10. $3x^2 - xy = 2,$ $2x^2 - 3xy + y^2 = 3.$

11. $x^2 + 2xy = 5,$ $x^2 - xy + y^2 = 3.$

12. $2x^2 + 5y^2 = 22,$ $x^2 + xy + 2y^2 = 11.$

Find to two significant figures the solutions of the following:

13. $x^2 + 0.12xy = 104,$ $y^2 + 1.4xy = 21.$

14. $xy = 2.3,$ $x^2 + 1.6y^2 = 8.3.$

15. Find two positive numbers such that their sum multiplied by the greater is 180, and their positive difference multiplied by the smaller is 27.

64. Symmetrical equations. An equation is said to be **symmetrical with respect to x and y** whenever interchanging x and y leaves the equation unchanged. The typical form of a symmetrical quadratic equation in two unknowns is

$$A(x^2 + y^2) + Bxy + D(x + y) + F = 0.$$

The solution is illustrated in the following.

Example: Solve

$$x^2 + y^2 + x + y = 8, \quad (1)$$
$$xy + x + y = 5. \quad (2)$$

Solution: Let $x = u + v$, $y = u - v$. Substituting in the two equations we obtain after reductions

$$u^2 + v^2 + u = 4, \tag{3}$$

$$u^2 - v^2 + 2u = 5. \tag{4}$$

Eliminating v^2 by adding we obtain an equation in u,

$$2u^2 + 3u = 9,$$

from which $u = \frac{3}{2}$ or -3.

The four solutions of (3) and (4) are then

$$\left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, -\frac{1}{2}\right), (-3, i\sqrt{2}), (-3, -i\sqrt{2}).$$

From the first pair

$$x = u + v = \frac{3}{2} + \frac{1}{2} = 2,$$

$$y = u - v = \frac{3}{2} - \frac{1}{2} = 1.$$

In a similar way we find from the other pairs,

$$\left.\begin{matrix} x = 1 \\ y = 2 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -3 + i\sqrt{2} \\ y = -3 - i\sqrt{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x = -3 - i\sqrt{2} \\ y = -3 + i\sqrt{2} \end{matrix}\right\}.$$

EXERCISES

Solve the following systems of equations:

1. $xy = 6,$
   $x^2 + y^2 = 13.$
2. $x^2 + y^2 - 13 = 0,$
   $xy - 2x - 2y + 4 = 0.$
3. $(x + y)^2 + x + y = 2,$
   $2xy + x + y + 3 = 0.$
4. $x^2 + y^2 = 30,$
   $3x^2 + 4xy + 3y^2 = 54.$
5. $s + t + 2st + 4 = 0,$
   $s^2 + t^2 + 2s + 2t = 8.$
6. $x^2 + y^2 - x - y - 22 = 0,$
   $x^2 + xy + y^2 + 2x + 2y = 29.$

65. Special devices. Many systems of equations of degree higher than two, and systems containing three or more unknowns may be solved by combinations and variations of the above methods, but these methods do not by any means apply to all the simultaneous equations whose solution can be reduced to the solution of the quadratic. Usually the solution of a system of this kind is such that special devices should be employed. Whatever method may be used it must be kept in mind that the ultimate test of a solution is substitution in the given equations. Many equations coming under the types solved in Arts. 60–64 may be solved more easily by other special methods.

For example, exercise 1 in Art. 64 may be solved as follows:

$$x^2 + y^2 = 13,$$
$$xy = 6.$$

Multiply the second equation by 2, add this result to and subtract it from the first equation, and thus obtain the system

$$x^2 + 2xy + y^2 = 25,$$
$$x^2 - 2xy + y^2 = 1.$$

Extracting the square root of each, we have

$$x + y = \pm 5,$$
$$x - y = \pm 1.$$

From these equations we find the same four results which were obtained by the general method for solving symmetrical equations.

In the exercises which follow, a number of such devices are suggested.

MISCELLANEOUS EXERCISES AND PROBLEMS INVOLVING QUADRATICS

Solve the following systems of equations. Check your results in exercises 1–5 by means of graphs.

1. $x + y = 3,$
$x^2 + y^2 = 29.$

2. $xy + 6 = 0,$
$x - y = 5.$

3. $xy = 16,$
$y^2 = 4x.$

4. $x^2 - y^2 = 15,$
$9x^2 + 16y^2 = 160.$

5. $4x^2 + 9y^2 = 36,$
$4x^2 = 9y^2.$

6. $x^2 + xy + y^2 = 28,$
$x^3 - y^3 = 56.$

Hint: Divide second by first.

7. $x^2 - xy + y^2 = 27,$
$2x^2 - 5xy + 2y^2 = 0.$

8. $\dfrac{1}{xy} + 28 = 0,$
$\dfrac{1}{x} - \dfrac{1}{y} = 11.$

Hint: Let $\dfrac{1}{x} = z, \dfrac{1}{y} = w;$ find z and $w;$ then find x and $y.$

9. $\dfrac{1}{x} - \dfrac{1}{y} = 5,$
$\dfrac{1}{x^2} + \dfrac{1}{y^2} = 233.$

10. $x^2 + 2xy - 5y^2 = 1,$
$2x^2 - 3xy + 4y^2 = 16.$

11. $xy = 2,$
$x^2 + y^2 = 4.$

12. $x - y = 4,$
$x^3 - y^3 = 988.$

Hint: Divide second by first.

13. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 34,$
$\dfrac{1}{xy} = 15.$

14. $\dfrac{16}{x^2} + \dfrac{27}{y^2} = 576,$
$\dfrac{1}{x^2} + \dfrac{1}{y^2} = 25.$

15. $2x^2 + 7xy - 15y^2 = 0,$
$(y - 2x - 1)(y - x + 2) = 0.$

Hint: Reduce to four linear systems by factoring the left side of the first equation.

16. $m^2 + 2n^2 = 9,$
$n^2 + nm = 2.$

17. $xy = 6,$
$x^2 + 4y^2 = 24.$

Hint: Multiply the first equation by 4, add to second, and solve for $x + 2y$.

18. $\frac{x^2}{y^2} + \frac{4x}{y} = \frac{85}{9},$
$x - y = 2.$

Hint: Solve first for $\frac{x}{y}$.

19. $xy = 6,$
$x^2 + y^2 + 3x + 3y = 28.$

Hint: Multiply first by 2, add to second, and solve for $x + y$.

20. $xy - 2y^2 = 0,$
$2x^2 + y^2 = 9.$

21. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1,$
$y = x.$

22. $x^{\frac{1}{2}} - y^{\frac{1}{2}} + 1 = 0,$
$y - 4x^{\frac{1}{2}} = 0.$

Hint: Let $x^{\frac{1}{2}} = z$ and $y^{\frac{1}{2}} = w$; find z and w; then find x and y.

23. $x^{\frac{1}{2}}y^{\frac{1}{2}} - 5x^{\frac{1}{2}} = 1,$
$7x^{\frac{1}{2}} - y^{\frac{1}{2}} = 1.$

24. $x^{\frac{1}{2}} - y^{\frac{1}{2}} = 1,$
$y = 4x^{\frac{1}{2}}.$

25. $x^2 - y^2 = 3,$
$x - y = 1.$

Hint: Divide first by second.

26. $x^3 + y^3 = 28,$
$x + y = 4.$

Hint: Divide first by second.

27. $x^{-2} - y^{-2} = 6,$
$x^{-1} - y^{-1} = 2.$

Hint: Let $x^{-1} = z$, $y^{-1} = w$; find z and w; then find x and y.

28. $x = y^{-1},$
$2x + y^{-1} = 6.$

29. $6xy = 1,$
$x^{-2} + y^{-2} = 13.$

30. $x^{-1} - y^{-1} = 2,$
$x^{-2} + y^{-2} - x^{-1}y^{-1} = 12.$

31. $x + y = 6,$
$x^2y^2 + 4xy = 96.$

Hint: Solve second for xy.

32. $2x^2 - xy = 10,$
$3x^2 - y^2 = 11.$

33. $x + y = 13,$
$\sqrt{x} - \sqrt{y} = 1.$

34. $x + \sqrt{y} = 7,$
$-\sqrt{x} + \sqrt{y} = 1.$

35. $\frac{4}{y^2} - \frac{1}{x^2} = 8,$
$\frac{3}{x^2} - \frac{4}{y^2} = 24.$

Hint: Solve first for $\frac{1}{x^2}$.

36. $x + 4y - z = 5,$
$2x + y + z = 6,$
$x^2 + 2y^2 - z^2 = 5.$

37. $x(y + z) = 8,$
$y(x + z) = 18,$
$z(x + y) = 20.$

Hint: Let $y = mx$, $z = nx$.

38. The sum of two numbers is 16 and their product is 63. Find the numbers.

39. The sum of two numbers is 200. The sum of their squares is 21,800. Find the numbers.

40. The sum of the reciprocals of two positive numbers is $\frac{5}{24}$, and the product of the numbers is 96. Find the numbers.

41. The sum of the squares of the two digits of a positive integral number is 89. The number itself is 7 more than 6 times the sum of its digits. Find the number.

42. The perimeter of a rectangle is 82 inches. Its area is 364 square inches. Find its length and breadth.

43. A rectangular farm of 120 acres has a diagonal of 200 rods. (An acre equals 160 square rods.) Find the sides.

44. Show that the formulas for the length l and width w of a rectangle in terms of its area A and diagonal d are

$$l = \frac{1}{2}[(d^2 + 2A)^{\frac{1}{2}} + (d^2 - 2A)^{\frac{1}{2}}], \quad w = \frac{1}{2}[(d^2 + 2A)^{\frac{1}{2}} - (d^2 - 2A)^{\frac{1}{2}}].$$

45. Show that the formulas for the length l and width w of a rectangle in terms of its perimeter p and area A are

$$l = \frac{1}{4}[p + (p^2 - 16A)^{\frac{1}{2}}], \quad w = \frac{1}{4}[p - (p^2 - 16A)^{\frac{1}{2}}].$$

46. A rectangular field contains 9 acres. If its length were decreased by 20 rods and its width by 4 rods, its area would be less by 4 acres. Find the length and width.

47. If the length of a diagonal of a rectangular field of 30 acres is 100 rods, how many rods of fence will be required to inclose the field?

48. It took a number of men as many days to pave a street as there were men, but had there been five more workmen employed, the work would have been done 4 days sooner. How many men were employed?

49. The sum of the squares of two consecutive integers is 1301. Find the numbers.

50. A rectangular piece of tin containing 400 square inches is made into an open box, containing 384 cubic inches, by cutting out a 6-inch square from each corner of the tin and then folding up the sides. Find the dimensions of the original piece of tin.

51. If the product of two numbers is increased by their sum, the result is 79. If their product is diminished by their sum, the result is 47. Find the numbers.

52. The hypotenuse of a right triangle is 100. If the shorter leg be increased by 30 and the longer by 40, the hypotenuse would be 150. Find each of the legs.

53. A rope 56 feet long exactly surrounds an inclosure in the form of a right triangle whose hypotenuse is 25 feet. Find the other sides of the inclosure.

54. Psychologists assert that the rectangle most pleasing to the human eye is that in which the sum of the two dimensions is to the longer as the longer is to the shorter. If the area of a page of this algebra remains unchanged, what should its dimensions be?

55. Two polygons have together 16 sides and 41 diagonals. How many sides has each?

56. An aëroplane, flying 75 miles per hour and following a long straight road, passed an automobile going in the opposite direction. One hour later it overtook a second automobile. The automobiles passed each other when the aëroplane was 100 miles away. If both automobiles traveled with the same speed, how far apart were they when the aëroplane passed the second one and what was their speed?

57. Two students attempt to solve a problem that reduces to a quadratic equation. One in reducing has made a mistake only in the constant term of the equation, and finds 8 and 2 for the roots. The other makes a mistake only in the coefficient of the first degree term, and finds -9 and -1 for roots. What was the quadratic equation?

58. A farmer raised broom corn and pressed 6120 pounds into bales. If he had made each bale 20 pounds heavier, he would have had one bale less. How many bales did he press and what was the weight of each?

59. After a mowing machine had made the circuit of a 10-acre rectangular field 33 times, cutting a swath 5 feet wide each time, $2\frac{1}{2}$ acres of grass were still standing. Find the dimensions of the field.

60. A club of boys bought a motorboat for $192. Four boys failed to pay their share as agreed, so each of the others was compelled to pay $4 more than he had promised. How many boys were in the club?

61. A father divided $2000 between his two sons and kept it for them at simple interest until called for. At the end of 3 years, one son called for all the money due him and received $1331. At the end of 4 years the other son received $1152 as his share. How was the money originally divided and what rate of interest did the father pay?

62. A few days after the outbreak of the war in 1914 a 25-pound bag of sugar cost the retailer $77\frac{1}{2}$ cents more than it did just before the outbreak. For $165 a grocer received 1550 pounds less sugar after the outbreak than he would have received before for the same amount. What was the price before and after the outbreak of the war?

63. The diagonal of a rectangular parallelopiped is 14 inches long. The sum of the three dimensions is 22. The reciprocal of one dimension is one half the sum of the reciprocals of the other two. What are the dimensions of the solid?

64. The diagonals of the three faces of a rectangular parallelopiped which meet in a vertex of the solid are 5, 6, 7, respectively. What is the volume of the solid?

CHAPTER IX

INEQUALITIES

66. Definition. The expressions "a is greater than b" ($a > b$) and "c is less than d" ($c < d$), when a, b, c, d, are real numbers, mean that $a - b$ is a positive number and $c - d$ is a negative number. Such expressions are called **inequalities.** Two inequalities $a > b$, $c > d$, which have the signs pointing in the same direction,

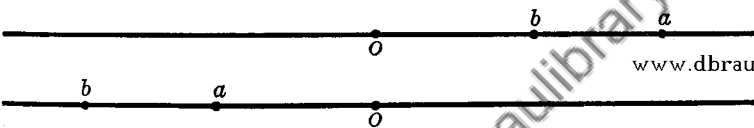

FIG. 22

are said to be **alike in sense.** If the signs point in opposite directions, as $a > b$, $c < d$, they are said to be **different in sense.** The expression $a \leqq b$ is read "a is less than or equal to b," and $a \geqq b$ is read "a is greater than or equal to b."

If the numbers are plotted on a straight line as in Fig. 22, then the statement $a > b$ means that a lies to the right of b.

67. Absolute and conditional inequalities. We have seen that there are two kinds of equalities, identical and conditional equalities. Corresponding to these there are two kinds of inequalities. An inequality such as $a^2 + b^2 > -1$, which is valid for all real values of a and b, is called an **absolute** inequality; while an inequality such as $x - 4 > 0$, which holds only when x is greater than 4, is called a **conditional** inequality. In a conditional inequality the letters cannot take all real values.

68. Elementary principles. The following elementary principles, which follow at once from the definition of an inequality, must be observed in dealing with inequalities.

I. *The sense of an inequality is not changed if both sides are increased or decreased by the same number. In particular, the sense is not changed if we transpose a term, changing its sign.*

Let $$a > b.$$

Then $\quad a - b = n$, where n is positive,

and $\quad a + k - b - k = n,$

or $\quad (a + k) - (b + k) = n.$

Hence, $\quad a + k > b + k.$

Examples: $7 > 3$, hence $7 + 6 > 3 + 6$, or $13 > 9$.

$-5 < -2$, hence $-5 + 10 < -2 + 10$, or $5 < 8$.

II. *The sense of an inequality is not changed if both sides are multiplied or divided by the same positive number.*

Examples: $7 > 3$, hence $7 \cdot 6 > 3 \cdot 6$, or $35 > 18$.

$-5 < -2$, hence $-5 \cdot 6 < -2 \cdot 6$, or $-30 < -12$.

III. *The sense of an inequality is reversed if both sides are multiplied or divided by the same negative number.*

Examples: $7 > 3$, hence $7 \cdot (-6) < 3 \cdot (-6)$, or $-42 < -18$.

$-5 < -2$, hence $-5 \cdot (-10) > -2(-10)$, or $50 > 20$.

$8 < 16$, hence $\dfrac{8}{-2} > \dfrac{16}{-2}$, or $-4 > -8$.

The proofs of II and III are very similar to the proof of I.

IV. *The sense of an inequality is reversed if each side is replaced by its reciprocal.*

Examples: $7 > 3$, hence $\dfrac{1}{7} < \dfrac{1}{3}$.

$-5 < -2$, hence $-\dfrac{1}{5} > -\dfrac{1}{2}$.

EXERCISES

1. Which of the above elementary principles justifies the statement that a term appearing on both sides of an inequality may be cancelled by subtracting the term from both sides?

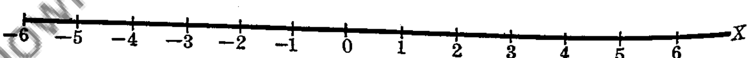

FIG. 23

2. How may terms be transposed from one side of an inequality to the other?

3. What part of the scale in Fig. 23 is included in the statement

$$-1 \leqq x \leqq 5?$$

4. What part of the scale in Fig. 23 is included in the statement $|x| \leqq 4$?

NOTE. The symbol $|x|$ is usually used to denote the numerical value of x, that is, the value without regard to sign. The expression $|x|$ is often read "absolute value of x."

5. If a and b are not equal, show that $a^2 + b^2 > 2ab$.

Solution: $(a - b)^2 > 0$, since the square of any real number is positive.

That is, $$a^2 - 2ab + b^2 > 0.$$

By Principle I, $$a^2 - 2ab + b^2 + 2ab > 0 + 2ab,$$

or $$a^2 + b^2 > 2ab.$$

6. Show that $\dfrac{a+b}{2} > \sqrt{ab}$, if a and b are positive and unequal.

This inequality states that the arithmetic mean of two positive numbers is greater than the geometric mean.

In exercises 7 to 12 the letters represent unequal positive numbers.

7. Show that $\dfrac{a}{b} + \dfrac{b}{a} > 2$.

8. The harmonic mean of two numbers, a and b, is $\dfrac{2ab}{a+b}\cdot$ Show that the harmonic mean of two numbers is always less than the geometric mean.

9. Given $a^2 + b^2 = 1$, $c^2 + d^2 = 1$. Show that $ab + cd < 1$.

Hint: $(a - b)^2 = a^2 + b^2 - 2ab$.

10. If $\dfrac{a}{b} < \dfrac{c}{d}$, show that $\dfrac{a}{b} < \dfrac{a+c}{b+d} < \dfrac{c}{d}$.

Hint: $\dfrac{a}{b} + \dfrac{c}{b} < \dfrac{c}{d} + \dfrac{c}{b}$, whence $\dfrac{a+c}{b+d} < \dfrac{c}{d}$.

11. Show that $\dfrac{a-b}{a+b} < \dfrac{a^2-b^2}{a^2+b^2}$, if $b < a$.

12. Show that $a + \dfrac{1}{a} > 2$ if a is positive and $\neq 1$.

13. Prove $|a| + |b| \geqq |a + b|$. (See exercise 4.)

Hint: Consider two cases, first when a and b have the same sign, second when they have different signs.

14. Prove $|a| + |b| \geqq |a - b|$.

15. Prove $|a| - |b| \leqq |a - b|$.

69. Conditional inequalities. By transposing terms every inequality may be reduced to an inequality of the form $P > 0$, or $P < 0$. If one or both sides involves a variable, say x, it can be put in one of the two forms $f(x) > 0$, or $f(x) < 0$. In this connection the most important problem is to find the range of values of the variable for which the inequality holds. In the case of linear inequalities the solution is easy. Thus, to find the values of x for which the inequality

$$3x + 19 > 12 - x$$

holds, all the terms can be transposed to the left-hand side, and there results

$$4x + 7 > 0.$$

Hence the inequality in question holds only for $x > -\frac{7}{4}$.

Graphically,

$$3x + 19 > 12 - x$$

for those values of x for which the graph of the function

$$3x + 19 - 12 + x \equiv 4x + 7$$

lies above the X-axis (Fig. 24).

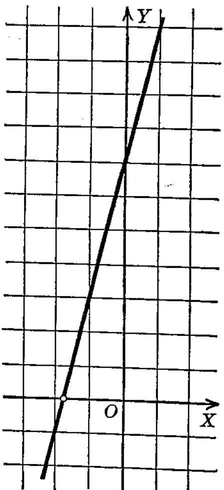

FIG. 24

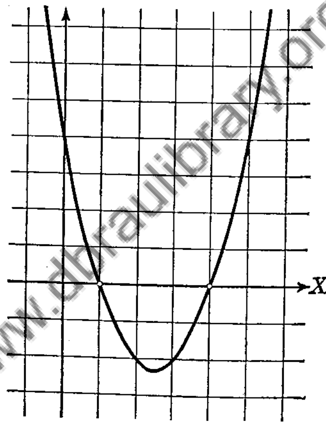

FIG. 25

The graph is of great service in determining the values of x for which one function of x is greater or less than another function. Thus, to find the range of values of x for which

$$2x^2 - 3x + 8 > x^2 + 2x + 4,$$

we transpose all terms to one side and have

$$x^2 - 5x + 4 > 0.$$

The graph of this function is shown in Fig. 25. It crosses the X-axis at 1 and 4, and for $x > 4$, or $x < 1$, the function $x^2 - 5x + 4$ is positive; while for $4 > x > 1$, it is negative; hence,

$$2x^2 - 3x + 8 > x^2 + 2x + 4$$

for $x > 4$, and $x < 1$, while

$$2x^2 - 3x + 8 < x^2 + 2x + 4$$

for $4 > x > 1$.

EXERCISES

For what values of x do the following inequalities hold?

1. $2x - 5 > 7.$

2. $2x - 5 > -7.$

3. $2x - 5 > 5x - 2.$

4. $2x - 5 > \dfrac{2x - 5}{2}.$

5. $3x + 4 < 5x + 6.$

6. For what values of x is $\dfrac{x+1}{x+2}$ negative?

7. $x^2 - x - 2 > 0.$

8. $2x^2 + x > 3.$

9. $13x - 6x^2 > 5.$

10. $3x^2 - 5x \begin{cases} > 2 \\ = 2 \\ < 2 \end{cases}$

11. $\dfrac{1}{x} > 1 + 2x.$

12. $|x| > \dfrac{1}{x}.$

13. For what values of x is

$$(x - 3)(2x + 1) < (x - 2)(3x + 1)?$$

14. For what values of k are the roots of the quadratic equation

$$4x^2 + 12x + k = 0$$

real and unequal? Imaginary?

Hint: See Art. 55.

15. Same as exercise 14 but for the quadratic equation

$$x^2 + kx + 4 = 0.$$

For what values of x do the following inequalities hold?

16. $\begin{vmatrix} 2x & 3 \\ 4 & 5 \end{vmatrix} > 0.$

17. $\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix} < \begin{vmatrix} x & 2 \\ 2 & 2 \end{vmatrix}.$

18. $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & x \end{vmatrix} < 1.$

19. $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & x \end{vmatrix} > \begin{vmatrix} x & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & x \end{vmatrix}.$

20. $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & x \end{vmatrix} > \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix}.$

CHAPTER X

RATIO, PROPORTION, AND VARIATION

70. Ratio. The **ratio** of a number a to a number b is the quotient $\frac{a}{b}$ obtained by dividing a by b. The ratio a to b is also written $a : b$.

It is clear from the above definition that any ratio is a fraction and any fraction may be regarded as a ratio. Thus, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{c}{d}$ are ratios.

71. Ratios involved in measurement. It is good usage and is often convenient to speak of the ratio of two quantities if they have a common unit of measure. Thus, the ratio of 6 feet to 2 feet is $\frac{6}{2}$.

To measure a quantity is to find its ratio to a given unit of measure. Thus, when we say a bar is 3 yards long, we mean that the ratio of the length of this bar to that of the standard yard is 3.

72. Proportion. A **proportion** is a statement of the equality of two ratios. Thus,

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion and is often written

$$a : b = c : d.$$

It is read "a is to b as c is to d."

The four numbers a, b, c, and d are said to be in proportion, a and d being called the **extremes** and b and c the **means** of the proportion.

EXERCISES

Find the value of x in the following proportions:

1. $\frac{x}{5} = \frac{7}{10}$.
2. $\frac{5}{x} = \frac{7}{10}$.
3. $x : 12 = 2 : 3$.
4. $\frac{3}{4} = \frac{x}{16}$.
5. $\frac{3}{5} = \frac{12}{x}$.
6. $4 : 7 = x : 14$.

7. If $\frac{a}{b} = \frac{c}{x}$, then x is said to be a **fourth proportional** to a, b, and c. Find a fourth proportional to the following sets of numbers:

(a) 5, 10, − 6.
(b) 4, 3, − 12.
(c) 4, − 6, − 5.

8. If $\frac{a}{x} = \frac{x}{d}$, then x is said to be a **mean proportional** between a and d. Find the mean proportional between the following sets of numbers:

(a) 4 and 9.
(b) + 3 and + 48.
(c) − 3 and − 48.

9. If $\frac{a}{b} = \frac{b}{x}$, x is said to be a **third proportional** to a and b. Find a third proportional to the following pairs of numbers:

(a) 2, 3.
(b) 2, − 3.
(c) 7, 11.
(d) − 7, − 11.

Given the proportion, $\frac{a}{b} = \frac{c}{d}$, prove the following:

10. $ad = bc$; product of means equals product of extremes.

11. $\frac{b}{a} = \frac{d}{c}$; said to be obtained by **inversion.**

12. $\frac{a}{c} = \frac{b}{d}$; said to be obtained by **alternation.**

13. $\frac{(a - b)}{b} = \frac{(c - d)}{d}$; said to be obtained by **division.**

14. $\frac{a + b}{a - b} = \frac{c + d}{c - d}$; obtained by **composition and division.**

PROBLEMS INVOLVING SIMILAR FIGURES

15. The sides of a triangle are 3, 4, and 5. In a similar triangle the shortest side is 5. What are the other sides? (Fig. 26.)

Similar figures are figures of the same shape. (Figs. 26, 27.) In two similar figures any two of the sides of one are proportional to the two corresponding sides of the second. The areas of similar figures have the same ratio as the squares of corresponding sides.

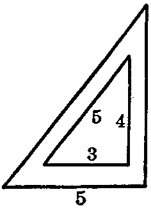

FIG. 26

16. The sides of a triangle are 9, 10, and 11. If the shortest side is lengthened one inch, what are the increases in the length of the other two sides in order to make the new triangle similar to the old?

17. In Fig. 27 the sides of the larger figure are 1, 2, 3, 4, 5. The longest side of the smaller figure is 3. What are the lengths of the other sides?

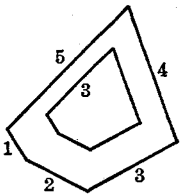

FIG. 27

18. The area of a triangle whose base is 12 inches is 60 square inches. If the area of a second triangle similar to the first is 135 square inches what is the base of the second triangle?

19. A post 4 feet high, 20 feet from a street light, casts a shadow 7 feet long. What is the height of the light on the lamp post?

20. If in a map the distance between two points 450 miles apart is 4 inches, what is the distance between two cities which are $5\frac{1}{2}$ inches apart on the map?

73. Variation. In Chapter III we have seen that if y is a function of x, written

$$y = f(x),$$

then in general y changes when x changes. We may say that y varies when x varies, but the word "varies" has come to have a more restricted meaning when used in this connection. Each of the statements

"y varies as x,"
"y varies directly as x,"
"y is proportional to x,"
"y is directly proportional to x,"

means that y is equal to the product of x by a constant. That is,

$$y = kx.$$

The constant k is called the **constant of variation.**

The expression "y varies as x" is sometimes written

$$y \propto x.$$

The area of a circle varies as the square of its radius. That is,

$$A = kr^2,$$

if A represents the area and r the radius. With our restricted meaning of the word "varies," it is not correct to say that the area of a circle varies as the radius, for, in the equality

$$A = k \cdot r,$$

k is not a constant for different values of r.

If a train moves with a uniform speed, the distance s traversed varies as the time t. That is,

$$s = kt.$$

74. Inverse variation. Each of the statements

"y varies inversely as x,"
"y is inversely proportional to x,"

means that y is equal to the product of the reciprocal of x and a constant. That is,

$$y = \frac{k}{x}.$$

The volume of air in the cylinder of a bicycle pump varies inversely as the pressure on the piston. That is,

$$V = \frac{k}{p},$$

if V represents volume and p pressure.

75. Joint variation. The statement "z varies jointly as x and y" means that z is equal to the product of x, y, and a constant. That is,

$$z = kxy.$$

The distance which a train, moving with a uniform speed, travels varies jointly as the speed and the time, or

$$d = kvt,$$

where d is the distance covered, v the speed, and t the time. In this case $k = 1$, if v and d are measured with the same unit of length.

76. Combined variation. The statement "z varies directly as x and inversely as y" means that z varies jointly as x and the reciprocal of y. That is,

$$z = \frac{kx}{y}.$$

If T varies directly as x, directly as the square of y, inversely as w and inversely as the cube of v, we have

$$T = k\frac{xy^2}{wv^3}.$$

The attraction F of any two masses m_1 and m_2 for each other varies as the product of the masses and inversely as the square of the distance r between the two bodies. That is,

$$F = \frac{km_1m_2}{r^2}.$$

77. Comments on problems of variation. Problems of variation frequently arise in experimental work.

Illustration 1. The time of vibration, t, of a simple pendulum varies as the square root of its length, l. It is found by experiment that a pendulum 39.1 inches long makes one vibration per second. Find the time of vibration of a pendulum of length 13 feet.

Solution. The law of variation is $t = k\sqrt{l}$.

To find k, put $t = 1$, $l = 39.1$. Then $k = \dfrac{1}{\sqrt{39.1}}$.

Hence, the law may be written

$$t = \sqrt{\frac{l}{39.1}}.$$

To find t when $l = 13$ feet $= 156$ inches, we have

$$t = \sqrt{\frac{156}{39.1}} = 2^{-} \text{ seconds.}$$

Illustration 2. The safe load of a horizontal beam supported at both ends varies jointly as the breadth, b, and square of the depth, d, and inversely as the length, l. If a 2×6 white pine joist safely holds up 800 pounds, what is the safe load of a 2×8 joist of same length?

Solution. The law of variation may be written

$$L = \frac{kbd^2}{l}.$$

From the given data, k is determined by the relation $800 = \dfrac{2 \cdot 6^2 \cdot k}{l}$.

Thus,

$$k = \frac{800l}{2 \cdot 6^2}.$$

The required safe load, $L = \dfrac{2 \cdot 8^2 k}{l} = \dfrac{8^2}{6^2} \cdot 800 = 1422\frac{2}{9}$ lbs.

EXERCISES AND PROBLEMS

Write each of the statements in exercises 1 to 9 in the form of an equation, using k as a constant of variation. Determine k when sufficient data are given.

1. The volume, V, of a cube varies as the cube of its edge, e.

2. The area, A, of a circle varies as the square of its radius, r.

3. The volume, V, of a sphere varies as the cube of its radius, r.

4. The volume, V, of a gas at constant temperature varies inversely as the pressure, p.

5. The attraction, A, of two particles of matter varies inversely as the square of the distance, d, between them.

6. The height, h, of a column of mercury in a thermometer varies directly as the temperature, T.

7. The weight, w, of a body above the surface of the earth is inversely proportional to the square of its distance, s, from the center of the earth. If an experiment gives $w = 150$ pounds when $s = 4000$ miles, find w when $s = 8000$ miles.

8. z is directly proportional to x and inversely proportional to y. Experiment shows $z = 6$ when $x = 9$ and $y = 3$.

9. z varies directly as the product of x and y and inversely as w. Experiment gives $z = 10$ when $x = 6$, $y = 5$, and $w = 3$.

10. Divide 200 into three parts proportional to 3, 7, 10.

11. An estate of \$100,000.00 is divided into four parts proportional to 1, 2, 4, 8. What are the parts?

Problems Involving the Strength of Materials

12. Write in the form of an equation the law: The safe load w of a horizontal beam supported at both ends varies jointly as the breadth b and the square of the depth d and inversely as the length l between supports.

13. A beam 15 feet long, 3 inches wide, and 6 inches deep when supported at both ends can bear safely a maximum load of 1800 pounds. What is the safe maximum load for a beam of the same material 10 feet long, 2 inches wide and 4 inches deep? (See problem 12.)

14. What is the safe load for the second beam mentioned in problem 13 if it is turned so that the width is 4 inches and depth 2 inches?

15. Write in the form of an equation the law: The crushing load, L, of a solid square oak pillar varies directly as the fourth power of its thickness, t, and inversely as the square of its length, l.

16. If a four-inch oak pillar 8 feet high is crushed by a weight of 100 tons, what weight will crush a pillar half as high and 6 inches thick? (See problem 15.)

17. What weight will crush a four-inch oak pillar 4 feet high?

18. The deflection D of a rectangular beam of a fixed length varies inversely as the product of the breadth b and the cube of the depth d. Write this statement in the form of an equation.

19. In the formula

$$s = \frac{kbd}{l},$$

s denotes the strength of a rectangular beam, b, d, and l, the breadth, depth, and length, respectively, of the beam, and k is a constant. State the formula in words, using the terms of variation.

Problems Involving Motion

20. The number of feet a body falls varies directly as the square of the number of seconds occupied in falling. If the body falls 16.1 feet the first second, how many feet will it fall in 6 seconds?

21. How far will a body fall during the sixth second?

22. The velocity of a falling body at any time varies directly as the number of seconds occupied in falling. What is the velocity at the end of 6 seconds if the velocity at the end of the first second is 32.2 feet per second?

23. An object dropped from a balloon strikes the ground in 7 seconds. At what velocity does the object strike the ground and what is the height of the balloon when the object is dropped?

24. A wrench is dropped from an automobile at a height of 3 feet while the automobile is traveling at the rate of 70 miles an hour. How far does the automobile move while the wrench is falling?

25. The time for one vibration of a pendulum at a given place varies as the square root of the length of the pendulum. In Chicago a pendulum 4 feet long requires 1.1 seconds for a vibration. What is the time of vibration of a pendulum 1 foot long?

26. What is the length of a pendulum which vibrates every second at Chicago?

27. A weight is suspended by a wire 94 feet long. What is the time of one vibration at Chicago?

28. A pendulum supposed to vibrate every second registers 90,000 vibrations in 24 hours. How much must the pendulum be lengthened?

Problems Involving Pressure

29. The volume of a gas inclosed in a vessel varies inversely as the pressure upon it. Twenty-four cubic inches of air under a pressure of 100 pounds will have what volume when the pressure is decreased to 50 pounds?

30. If a toy balloon contains 150 cubic inches of gas when under a pressure of 15 pounds per square inch, to what size will it shrink if subjected to a pressure of 45 pounds per square inch? (See problem 29.)

31. The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the velocity is 15 miles per hour, the pressure on a square foot is 1 pound. What is the velocity of the wind when the pressure is 10 pounds per square foot?

32. The pressure of gas in a tank varies jointly as its density and its absolute temperature. When the density is 1 and the temperature 300°, if the pressure is 15 pounds per square inch, what is the pressure when the density is 2 and the temperature 290°?

CHAPTER XI

PROGRESSIONS

78. Arithmetic progressions. An **arithmetic progression** is a sequence of numbers each of which differs from the next preceding one by a fixed number called the **common difference.** Thus,

$$2, 4, 6, 8, \cdots$$

is an arithmetic progression with the common difference 2. In the arithmetic progression

$$10, 8, 6, 4, 2, \cdots$$

the common difference is -2.

The numbers of the sequence are called the **terms** of the progression.

79. Elements of an arithmetic progression. Let a represent the first term, d the common difference, n the number of terms considered, l the nth, or last term, and s the sum of the sequence. The five numbers a, d, n, l, and s are called the **elements** of the arithmetic progression.

80. Relations among the elements. Since a is the first term, we have, by definition of an arithmetic progression,

$$\begin{aligned} a + d &= \text{second term}, \\ a + 2d &= \text{third term}, \\ a + 3d &= \text{fourth term}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ a + (n - 1)d &= n\text{th term}. \end{aligned}$$

That is,
$$\boldsymbol{l = a + (n - 1)d.} \tag{1}$$

The sum of an arithmetic progression may be written in each of the following forms:

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l,$$
$$s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

By addition

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l)$$
$$= n(a + l).$$

Therefore,

$$s = \frac{n}{2}(a + l). \tag{2}$$

Whenever any three of the five elements are given, equations (1) and (2) make it possible to find the remaining two elements.

81. Arithmetic means. The first and last terms of an arithmetic progression are called the **extremes,** while the remaining terms are called the **arithmetic means.** To insert a given number of arithmetic means between two numbers it is only necessary to determine d by the use of equation (1) and to write down the terms by the repeated addition of d.

EXERCISES

Continue each of the following sequences to three additional terms.

1. $2, -1, -4, -7, \cdots$.

Solution: Each term in this sequence may be obtained by adding -3 to the preceding term. Hence, the sequence is an arithmetic progression where $d = -3$, and three additional terms are $-10, -13, -16$.

2. $1, 5, 9, 13, \cdots$.

3. $-2, 5, 12, 19, \cdots$.

4. $\frac{2}{3}, \frac{7}{12}, \frac{1}{2}, \frac{5}{12}, \cdots$.

Find l and s for the following sequences in exercises 5 to 9:

5. $2, 11, 20, \cdots$ to 10 terms.

Solution:
$$l = a + (n - 1)d.$$

Here,
$$a = 2,\ d = 9,\ n = 10.$$
$$l = 2 + 9 \cdot 9 = 83.$$
$$s = \frac{n}{2}(a + l)$$
$$= 5(2 + 83) = 425.$$

6. $1, 3, 5, 7, \cdots$ to 11 terms.

7. $2, 6, 10, 14, \cdots$ to 12 terms.

8. $5, 1, -3, -7$, to 20 terms.

9. $3, \frac{3}{2}, 0, -\frac{3}{2}$, to 8 terms.

10. Given $a = 3$, $d = 6$, $s = 363$; find n and l.

11. Given $a = -6$, $n = 12$, $l = 72$; find d and s.

12. Given $d = -2$, $n = 10$, $l = -17$; find a and s.

13. Given $n = 11$, $l = 20$, $s = 0$; find a and d.

14. Given $a = 7$, $n = 7$, $s = 7$; find d and l.

15. Find the sum of the odd numbers less than 100.

16. Find the sum of the odd numbers less than x, where x is an even number.

17. Insert six arithmetic means between 3 and 8.

Solution: We have to find d, when $a = 3$, $l = 8$, and $n = 6 + 2 = 8$.

Since $$l = a + (n - 1)d,$$

we have $$8 = 3 + 7d, \text{ or } d = \frac{5}{7}.$$

Hence, the six arithmetic means between 3 and 8 are

$$\frac{26}{7}, \frac{31}{7}, \frac{36}{7}, \frac{41}{7}, \frac{46}{7}, \frac{51}{7}.$$

18. Insert three arithmetic means between 4 and 16.

19. Insert five arithmetic means between 4 and 18.

20. Find the arithmetic mean between $8\frac{1}{2}$ and $17\frac{1}{2}$.

21. Insert two arithmetic means between x and y.

22. Insert two arithmetic means between x and x^2.

82. Geometric progressions. A **geometric progression** is a sequence of numbers in which the same quotient is obtained by dividing any term by the preceding term. This quotient is called the **common ratio.** Thus,

$$3, 6, 12, 24, \cdots$$

is a geometric progression with a common ratio 2.

83. Elements of a geometric progression. The elements are the same as those for an arithmetic progression with one exception. Instead of the common difference of an arithmetic progression, we have here a common ratio represented by r.

84. Relations among the elements. If a represents the first term, then

$$ar = \text{second term},$$
$$ar^2 = \text{third term},$$
$$ar^3 = \text{fourth term},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$ar^{n-1} = n\text{th term}.$$

That is, $$l = ar^{n-1}. \tag{1}$$

By definition,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \tag{2}$$

Then, $\quad sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (3)$

Subtracting members of (2) from members of (3), we have

$$sr - s = ar^n - a.$$

Hence, $\quad s = \dfrac{ar^n - a}{r - 1} = a\dfrac{(1 - r^n)}{1 - r}. \qquad (4)$

Since $\quad l = ar^{n-1}$, (4) may be written in the form

$$s = \frac{rl - a}{r - 1}. \qquad (5)$$

Here, as in an arithmetic progression, whenever any three of the five elements are given, relations (1) and (5) make it possible to find the other two.

85. Geometric means. The first and last terms of a geometric progression are called the **extremes,** while the remaining terms are called the **geometric means.** To insert n geometric means between two given numbers is to find a geometric progression of $n + 2$ terms having the two given numbers for extremes.

EXERCISES

1. Write five terms of the geometric progression whose first term is $\frac{2}{3}$ and whose common ratio is $\frac{3}{2}$.

2. The first two terms of a geometric progression are $\frac{1}{2}$ and $\frac{1}{3}$. Write down the next three terms.

3. Find the eleventh term of the sequence $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

Solution:
$$a = 1, \quad n = 11, \quad r = \frac{1}{2}.$$
$$l = ar^{n-1} = 1 \cdot \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}.$$

4. Find the twelfth term of $16, 8, 4, \cdots$.

5. Given $a = 2$, $r = -3$, $n = 10$; find l and s.

6. Given $a = \frac{1}{8}$, $r = 2$, $n = 8$; find l and s.

7. Given $s = 13$, $a = 1$, $n = 3$; find r and l and check your result by writing out the progression.

8. The third term of the geometric progression is 3, and the sixth term is 81. What is the tenth term?

9. Find the sum of $3, 6, 12, \cdots, 192$.

10. Find l and s when $a = 7$, $r = 7$, $n = 7$.

11. Given $r = 7$, $n = 7$, $l = 7$, find a and s.

12. The fifth term of a geometric progression is 5, the common ratio is 3; find the tenth term.

13. The fourth term of a geometric progression is 0.9, the seventh is 0.0243. Find the first two terms.

14. Find the 13th term and the sum to thirteen terms of the sequence, $1, \sqrt{3}, 3, \cdots$.

15. Find the last term and the sum to seven terms of the sequence,

$$(2x - 9y),\ (7x + 3y),\ (12x - y),\ \cdots.$$

16. Insert two geometric means between 2 and 1024.

Solution: We have to find r, when $a = 2$, $l = 1024$, and $n = 4$.

Since $$l = ar^{n-1},$$

we have $$1024 = 2r^3 \text{ or } r^3 = 512,\ r = 8.$$

Hence, the two geometric means between 2 and 1024 are 16 and 128.

17. Insert one geometric mean between 5 and 3125.

18. Insert three geometric means between 81 and 1. Give two solutions.

19. The geometric mean between two numbers is 33. One of the numbers is 9. What is the other?

20. Insert three geometric means between $3a^2$ and $\frac{48}{a^2}$.

86. Number of terms infinite. Consider the geometric progression

$$\tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \cdots.$$

It may at first thought appear that the sum of the first n terms of this progression could be made to exceed any finite number previously assigned by making n large enough. That this is not

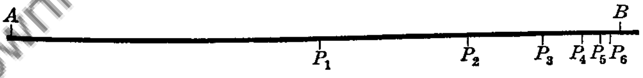

FIG. 28

the case and that the sum can never exceed unity, will be seen from the following illustration. Conceive a particle moving in a straight line towards a point one unit distant in such a way as to describe $\frac{1}{2}$ the distance in the first second, $\frac{1}{2}$ the remaining distance in the second second, $\frac{1}{2}$ the remaining distance in the third second, and so on indefinitely. This is represented in Fig. 28.

The distance AB represents one unit of distance. In the first second the particle moves from A to P_1. In the second second it moves from P_1 to P_2, and so on. The total distance traversed by the particle in n seconds is given by the sum

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \text{ to } n \text{ terms},$$

which sum cannot exceed nor equal 1, no matter how many terms we take, but can be made to differ from 1 by as small a positive number as we please by making the number of terms large enough. In this illustration, 1 is said to be the limiting * value of the sum of the first n terms of the series. If s_n represents the sum of the first n terms of the series, we write

$$s = \lim_{n \to \infty} s_n = 1,$$

which reads, "the limit of s_n as n increases **beyond bound** is 1." †

The limit s is called the **sum** of the geometric progression with infinitely many terms.

For any geometric progression in which the ratio is less than 1, the above argument can be repeated, and it can be shown that there is a limiting value to the sum of the first n terms of such a series. In Art. 84, we have shown that the sum of the geometric progression

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

is given by $$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

We may then write

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a}{1 - r} - \lim_{n \to \infty} \frac{ar^n}{1 - r}. \quad \text{(See Art. 182.)}$$

It will be proved in the chapter on Limits (Art. 184) that

$$\lim_{n \to \infty} \frac{ar^n}{1 - r} = 0 \text{ when } |r| < 1.$$

Hence, $$s = \lim_{n \to \infty} s_n = \frac{a}{1 - r}.$$

87. Series. A **series** is an expression which consists of the sum of a sequence of terms. Thus, the indicated sum of the terms of a progression is often called a series.

A **finite** series is one which has a limited number of terms.

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* For definition of "limit," see Art. 181.

† The symbol "$n \to \infty$" stands for "n increases beyond bound," or its equivalent "n becomes infinite."

An **infinite** series is one in which the number of terms is infinite; that is, the number of terms has no bound.

88. Repeating decimals. Repeating decimals furnish good illustrations of infinite series which are at the same time the sum of the terms of a geometric progression with infinitely many terms. For example, $0.33333 \cdots$ may be written as the series

$$0.3 + 0.03 + 0.003 + 0.0003 + \cdots,$$

where $a = 0.3$ and $r = 0.1$. The limit of the sum of n terms of this series as the number n increases indefinitely is $\frac{1}{3}$. Again, $0.9828282 \cdots$ may be written

$$0.9 + 0.082 + 0.00082 + \cdots,$$

where the terms after the first form a geometric progression in which $a = 0.082$ and $r = 0.01$.

The expression "limit of the sum of n terms of the infinite series as n increases beyond bound" is often abbreviated by saying merely **"sum of the infinite series."**

EXERCISES

Find the sum of the following progressions:

1. $1, \frac{1}{3}, \frac{1}{9}, \cdots$.

2. $1, -\frac{1}{3}, \frac{1}{9}, \cdots$.

3. $6, 4, 2\frac{2}{3}, \cdots$.

4. $75, 15, 3, \cdots$.

5. $49, -21, 9, \cdots$.

6. $8, 4\sqrt{2}, 4, \cdots$.

Find the limiting value of the following repeating decimals:

7. $0.909090 \cdots$.

8. $0.77777 \cdots$.

9. $0.70707 \cdots$.

10. $8.3707 \cdots$.

Hint: Write the number as $8.3 + 0.070707 \cdots$.

11. $0.741414 \cdots$.

12. $123.123123 \cdots$.

89. Harmonic progressions. Three or more numbers are said to form a **harmonic progression** if their reciprocals form an arithmetic progression. The term "harmonic" as here used comes from a property of musical sounds. If a set of strings of uniform tension whose lengths are proportional to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ be sounded together, the effect is harmonious to the ear. The sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \cdots$$

is a harmonic progression since the reciprocals form the arithmetic progression

$$1, 2, 3, 4, 5, \cdots.$$

90. Harmonic means. To find n harmonic means between two numbers, find n arithmetic means between the reciprocals of these numbers. The reciprocals of the arithmetic means are the harmonic means.

EXERCISES

1. Show that the sequence, $\frac{3}{7}, \frac{1}{3}, \frac{3}{11}, \frac{3}{13}, \frac{1}{5}$, is a harmonic progression.

2. Continue the harmonic progression, $1, \frac{5}{2}, -5, \cdots$, for three more terms.

3. Show that 2, 3, 6 are in harmonic progression, and continue the series for two terms in each direction.

4. Insert two harmonic means between $\frac{1}{3}$ and $\frac{1}{6}$.

5. Insert two harmonic means between 2 and $\frac{4}{5}$.

6. Insert four harmonic means between $\frac{1}{12}$ and $\frac{1}{2}$.

7. What is the harmonic mean between a and b?

PROBLEMS

1. A ball rolls down an incline 7.47 feet the first second, and in each succeeding second, 14.94 feet more than in the preceding second. How far will it roll in 10 seconds?

2. How many ancestors has a person in the ten preceding generations counting his two parents, four grandparents, eight great grandparents and so on (assuming no duplicates)?

3. What distance will an elastic ball traverse before coming to rest if it be dropped from a height of 60 feet and if after each fall it rebounds one sixth of the height from which it falls?

4. If a falling body descends $16\frac{1}{12}$ feet the first second, 3 times this distance the next, 5 times the next, and so on, how far will it fall the 30th second, and how far altogether in 30 seconds?

5. Assume that a baseball will fall 16 feet the first second, 48 the next, 80 the next, and so on. A baseball was dropped from the top of Washington Monument, 550 feet high, and caught by an American League catcher. About how fast was the ball falling when caught?

6. A swinging pendulum is brought gradually to rest by friction of the air. If the length of the first swing of the pendulum bob is 30 centimeters,

and the length of each succeeding swing is $\frac{1}{10}$ less than the preceding one, what is the distance passed over in the fifth swing?

7. What is the total distance passed over by the pendulum bob described in problem 6 in 5 swings?

8. A person contributes one cent and sends letters to four friends asking each to contribute one cent to a certain charity and to write a similar letter to four friends, each of whom is to write four letters, — and so on until nine sets of letters have been written. If all respond, how much money will the charity receive?

9. Twenty-five stones are placed in a straight line on the ground at intervals of 4 feet. A basket is placed 10 feet from the end of the row. A runner starts from the basket and picks up the stones and carries them, one at a time, to the basket. How far does he run altogether?

10. An employer hires a clerk for five years at a beginning salary of \$500 per year with either a raise of \$100 each year after the first, or a raise of \$25 every six months after the first half year. Which is the better proposition for the clerk?

11. What annual increase in salary is equal over a period of six years to a monthly increase of \$5.

12. What is the sum of the first n odd numbers?

13. What is the sum of the first n even numbers?

14. A rubber ball is dropped from a height of 120 inches. On each rebound the ball rises to $\frac{2}{3}$ of the height from which it last fell. Find the distance traveled by the ball before coming to rest.

15. Find the limiting value of the sum of the series

$$\frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} + \cdots \text{ where } x > 0.$$

16. If A, G, and H stand respectively for the arithmetic, geometric, and harmonic means between two numbers a and b, show that $G^2 = AH$.

17. The fourth term of a geometric progression is 81, the seventh is 9. What is the tenth term?

18. What is the equation whose roots are the arithmetic and the harmonic means between the roots of $x^2 - 16x + 48 = 0$?

19. If $\frac{1}{b-a}$, $\frac{1}{2b}$, $\frac{1}{b-c}$ form an arithmetic progression, show that a, b, and c form a geometric progression.

20. Find the common fraction equal to $.4919191 \cdots$.

21. Find the improper fraction equal to $4.037037037 \cdots$.

22. Given unequal positive numbers a and b, show that their arithmetic mean is greater than their positive geometric mean.

23. A pail containing 6 quarts of water was passed in succession to 11 football players. Each drank $\frac{1}{8}$ of the water that was in the pail when he received it. How much water was left in the pail after all had drunk?

24. The sides of a right triangle are in arithmetic progression. Show that the triangle is similar to the triangle whose sides are 3, 4, 5.

25. The sum of an arithmetic progression is 36, the first term, 15, and the common difference, -3. Find two values of n which satisfy these conditions.

26. The mth term of an arithmetic progression is M. The nth term is N. What is the first term and the common difference?

CHAPTER XII

MATHEMATICAL INDUCTION AND THE BINOMIAL THEOREM

91. Introduction. Many important theorems in algebra can be proved by a method called **mathematical induction.** The principle on which this method of proof depends may be illustrated as follows. Imagine a line of men at a ticket window seeking to purchase tickets for a football game. Suppose we could show that: (1) The first man in line obtains a ticket; (2) if anyone gets a ticket so does the next in line. We conclude from these propositions that everyone in the line obtains a ticket, regardless of the number of persons in the line. We begin the explanation of this method by applying it to a simple example.

Let it be required to show that $x^n - y^n$ is exactly divisible by $x - y$ for all positive integral values of n. If $n = 1$, we have $x - y$, which is divisible by $x - y$ with the quotient 1. This corresponds to condition (1). Let r be any value of n for which the proposition is true. Then

$$x^r - y^r = (x - y)Q, \tag{1}$$

where Q is a polynomial (Art. 24) in x and y. Since

$$\begin{aligned} x^{r+1} - y^{r+1} &= xx^r - yy^r = xx^r - xy^r + xy^r - yy^r \\ &= x(x^r - y^r) + y^r(x - y) \\ &= x(x - y)Q + (x - y)y^r \qquad \text{by (1)} \\ &= (x - y)(xQ + y^r), \end{aligned}$$

the proposition is true for $r + 1$ whenever it is true for r. This corresponds to condition (2).

Since the proposition holds for $n = 1$, it follows from the second part of the argument that it holds for the next integer, that is, for $n = 2$. Its validity for $n = 3$, 4, and so on follows for the same reasons.

It is no proof simply to show that a theorem is true in a number of cases. For example, the above theorem is not proved by showing that it is true for $n = 1$, $n = 2$, $n = 3$, $n = 4$, and so on for a definite number of cases. The second part of the proof is necessary.

A celebrated example illustrating this point is the expression $n^2 - n + 41$. From the table we see that $n^2 - n + 41$ is a prime

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
$n^2 - n + 41$	41	43	47	53	61	71	83	97	113	131	151	173

number for all integral values of n up to 12. The table could be continued up to $n = 40$ and the lower row would still contain nothing but prime numbers. However we have no proof that $n^2 - n + 41$ is prime for all integral values of n. To prove this it is necessary to take the second step in the proof by mathematical induction, i.e., to prove that if $r^2 - r + 41$ is prime, then $(r + 1)^2 - (r + 1) + 41$ is prime. But this is impossible. In fact, when $n = 41$, we have

$$n^2 - n + 41 = 1681 = 41^2,$$

a number which is not prime.

Again, it is no proof simply to show that if a statement is true for $n = r$ it is true for $n = r + 1$. For example, assuming that the sum of the first r even numbers is an odd number it follows that the sum of the first $r + 1$ even numbers is odd. Though the second part of the proof of this statement by mathematical induction can be correctly presented, we know the statement to be false. The first part of the proof is lacking.

We emphasize the two parts as follows:

Part I. To show by mere verification that the proposition in question is true for some particular case, usually for $n = 1$.

Part II. To show that if it is true for $n = r$, it is then true for $n = r + 1$.

If these two steps are completed it follows that the proposition is true for every positive integral value of n, equal to or greater than the one for which the verification was made in Part I. The verification for values of n beyond $n = 1$ may be satisfying but this is not relevant to the proof.

EXERCISES

Prove by mathematical induction, n being any positive integer.

1. $$2 + 4 + \cdots + 2(n - 1) + 2n = n(n + 1). \qquad (1)$$

Solution: Part I. If $n = 1$ the series on the left of equation (1) reduces to the single term 2. But $2 = 1(1 + 1)$, so the equation holds if $n = 1$.

Part II. Let r be any value of n for which equation (1) is true. That is,

$$2 + 4 + \cdots + 2(r - 1) + 2r = r(r + 1) = r^2 + r. \qquad (2)$$

We wish to show that the equation obtained from (1) by letting $n = r + 1$ is a consequence of equation (2). We are interested in the question: Is

$$2 + 4 + \cdots + 2r + 2(r + 1) = (r + 1)[(r + 1) + 1] = r^2 + 3r + 2? \qquad (3)$$

The left side of equation (3) contains all the terms of the left side of equation (2) and the single additional term $2(r + 1)$. Subtracting equation (2) from equation (3) we obtain

$$2(r + 1) = 2r + 2,$$

an identity. Thus equation (3) is implied by equation (2) and the proof of Part II is complete. Equation (1) is therefore valid for all positive integral values of n.

Remark. The proof in Part II can also be made by adding the $(r + 1)$th term of the progression to both sides of equation (2) and reducing the expression on the right to the form $(r + 1)(r + 2)$.

2. $1 + 2 + 3 + \cdots + n = \dfrac{n}{2}(n + 1).$

3. $1 + 3 + 5 + \cdots + (2n - 1) = n^2.$

4. $3 + 6 + 9 + 12 + \cdots + 3n = \dfrac{3n(n + 1)}{2}.$

5. $2 + 2^2 + \cdots + 2^n = 2(2^n - 1).$

6. $\dfrac{1}{2} + \dfrac{1}{4} + \cdots + \left(\dfrac{1}{2}\right)^n = 1 - \left(\dfrac{1}{2}\right)^n.$

7. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}.$

8. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{1}{6}n(n + 1)(2n + 1).$

9. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4} = (1 + 2 + \cdots + n)^2.$

10. $x^{2n} - y^{2n}$ is divisible by $x + y$.

11. $x^{2n+1} + y^{2n+1}$ is divisible by $x + y$.

12. $\dfrac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}.$

13. Establish the formulas for the last term and the sum of an arithmetical progression by mathematical induction.

92. Meaning of $r\,!$. The symbol $r\,!$, read "factorial r," * is used to indicate the product $1 \cdot 2 \cdot 3 \cdots r$. Thus, $3\,! = 1 \cdot 2 \cdot 3 = 6$; $7\,! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$.

* The symbol $\underline{|r}$ is often used to represent factorial r.

EXERCISES

Evaluate the following expressions:

1. $\frac{9!}{6!}$. 2. $\frac{8!}{7!}$. 3. $\frac{3!5!}{4!}$. 4. $\frac{3!+5!}{4!}$.

5. $\frac{9!}{3!6!}$. 6. Prove $\frac{r!}{(r-1)!} = r$. 7. $\frac{r!}{(r-2)!} = ?$

93. Binomial theorem; positive integral exponents. By multiplication, we find

$$(a + x)^2 = a^2 + 2ax + x^2.$$

$$\begin{aligned}(a + x)^3 &= a^3 + 3a^2x + 3ax^2 + x^3\\ &= a^3 + 3a^2x + \frac{3 \cdot 2}{2!} ax^2 + x^3.\\ (a + x)^4 &= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4\\ &= a^4 + 4a^3x + \frac{4 \cdot 3}{2!} a^2x^2 + \frac{4 \cdot 3 \cdot 2}{3!} ax^3 + x^4.\end{aligned}$$

If n represents the exponent of the binomial in any one of the above three cases, we notice:

(1) The first term is a^n.

(2) The second term is $na^{n-1}x$.

(3) The exponents of a decrease by unity from term to term while the exponents of x increase by unity.

(4) If in any term the coefficient be multiplied by the exponent of a and divided by the exponent of x increased by unity, the result is the coefficient of the next term.

For $n < 5$, we may then write

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!} a^{n-2}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!} a^{n-r+1}x^{r-1} + \cdots + x^n.$$

Here the question naturally occurs: Does the expansion hold for $n \geqq 5$? It can be shown by mathematical induction that it holds for any positive integral value of n.

Assume

$$(a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2!} a^{m-2}x^2 + \cdots + \frac{m(m-1)\cdots(m-r+2)}{(r-1)!} a^{m-r+1}x^{r-1} + \cdots + x^m.$$

Multiply both members of this assumed equality by $a + x$, and we obtain

$$
\begin{aligned}
&(a+x)^{m+1} = \\
&a^{m+1} + ma^m x + \cdots + \frac{m(m-1)\cdots(m-r+2)}{(r-1)!}a^{m-r+2}x^{r-1} + \cdots + ax^m \\
&\quad + a^m x + \cdots + \frac{m(m-1)\cdots(m-r+3)}{(r-2)!}a^{m-r+2}x^{r-1} + \cdots + max^m + x^{m+1} \\
\hline
&= a^{m+1} + (m+1)a^m x + \cdots + \frac{(m+1)m\cdots(m-r+3)}{(r-1)!}a^{m-r+2}x^{r-1} + \cdots \\
&\quad + (m+1)ax^m + x^{m+1}
\end{aligned}
$$

This expansion is the same that would be obtained by substituting $m + 1$ for m in the expansion of $(a + x)^m$. Hence, if the expansion is true for $n = m$, it is true for $n = m + 1$. Since we know it is true for $n = 2$, it is true for $n = 3$, and so on. Hence, when n is any positive integer,

$$
\begin{aligned}
(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots \\
+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1} + \cdots + x^n.
\end{aligned}
$$

This expansion of a binomial is called the **binomial theorem.**

94. The general term of $(a + x)^n$. In the expansion of $(a + x)^n$, the rth term is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1},$$

which may also be written

$$\frac{n!}{(r-1)!(n-r+1)!}a^{n-r+1}x^{r-1}.$$

The term involving x^r is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}x^r = \frac{n!}{r!(n-r)!}a^{n-r}x^r.$$

Each of these terms is sometimes called the **general term of the binomial expansion.**

In this chapter the exponent n in the binomial expansion is limited to positive integral values, but no assumption has been made with regard to a or x, so we are at liberty to use the expansion no matter what sort of numbers a and x may be. Thus, in $(2b - 4r^2)$, $a = 2b$ and $x = -4r^2$. In a later chapter it will be shown that the expansion may be interpreted so as to hold when n is a negative or fractional number, but in that case the number x must lie between $-a$ and $+a$.

EXERCISES

Expand the binomials in exercises 1–11, simplify, and check the result for the special case in which each letter is equal to 1.

1. $(2x - 3y^3)^4$.

Solution:

$$(2x - 3y^3)^4 = (2x)^4 + 4(2x)^3(-3y^3) + 6(2x)^2(-3y^3)^2 + 4(2x)(-3y^3)^3 + (-3y^3)^4 = 16x^4 - 96x^3y^3 + 216x^2y^6 - 216xy^9 + 81y^{12}.$$

Check:

$$(2-3)^4 = 16 - 96 + 216 - 216 + 81.$$
$$1 = 1.$$

2. $(a + x)^6$.

3. $(a - x)^6$.

4. $(a + b)^7$.

5. $(2 - x)^5$.

6. $\left(\frac{1}{2} + 2a\right)^4$.

7. $(1 + a)^5$.

8. $(x - 3y^2)^5$.

9. $(a + \sqrt{b})^6$.

10. $\left(x + \frac{1}{x}\right)^4$.

11. $\left(1 + \frac{1}{e}\right)^6 - \left(1 - \frac{1}{e}\right)^6$.

Expand by the binomial theorem and simplify:

12. $\begin{vmatrix} a & 1 \\ 2 & b \end{vmatrix}^4$.

13. $(\sqrt{x} - \sqrt{y})^8$.

14. $\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)^6$.

15. $(a^{\frac{3}{2}} + a^{\frac{2}{3}})^6$.

16. $(x^{-1} - y^{\frac{1}{3}})^4$.

17. $(3 - 1)^5$.

18. $\left(\frac{a}{\sqrt{b}} - \frac{b}{\sqrt{x}}\right)^6$.

19. $(a + b + c)^3$.

Hint: Consider $(a + b)$ as representing one number.

20. $\left(\sqrt{x} + 2 + \frac{1}{x}\right)^3$.

21. $\left(\frac{x^2}{3} - \frac{2}{x^3} + 1\right)^4$.

Find the first three terms of each expansion and simplify:

22. $(x^{\frac{1}{2}} - 3y^2)^{11}$.

23. $\left(a^{\frac{1}{2}} - \frac{2}{3}a^{-\frac{3}{2}}\right)^{13}$.

Find the last three terms of each expansion and simplify:

24. $\left(\frac{a^{\frac{1}{3}}}{2b} - a^{-\frac{1}{2}}\right)^{10}$.

25. $(x^{-\frac{1}{2}} + x^{\frac{1}{2}})^{15}$.

26. Find the seventh term in the expansion of $(x^{\frac{1}{2}} - 2y^2)^{11}$.

Solution: The rth term is given by the expression

$$\frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1}.$$

Here $n = 11, r = 7, a = x^{\frac{1}{2}}, x = -2y^2, n - r + 2 = 6.$

Substituting these in the expression for the rth term, we have

$$\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{6!}(x^{\frac{1}{2}})^5(-2y^2)^6 = 29568x^{\frac{5}{2}}y^{12}.$$

27. Find the fourth term of $(a - 4b)^{12}$.

28. Find the eleventh term of $(2x - y)^{17}$.

29. Find the middle term of $(x^2 + 3y^2)^8$.

30. Find the fourteenth term of $(a + b)^{18}$.

31. Find the eighth term of $\begin{vmatrix} x & a \\ \sqrt{a} & y \end{vmatrix}^{13}$.

32. Find the sixth term of $(x\sqrt{y} + y\sqrt{x})^9$.

33. Find the middle terms of $(1 - a^{\frac{2}{5}})^7$.

34. Use the binomial theorem to find $(102)^5$.

Hint: $102 = (100 + 2)$.

Use the binomial theorem to find

35. $(99)^6$. **36.** $(51)^5$. **37.** $(.98)^6$.

38. $(1.1)^{10}$, correct to four significant figures.

39. $(1.1)^{15}$, correct to four significant figures.

40. Number the terms of the binomial expansion $(1 + 1)^6$. With these numbers as abscissas and with the corresponding value of the terms as ordinates, plot points and connect the points plotted by straight lines. From this figure give a general description of the manner in which the successive terms increase and decrease.

CHAPTER XIII

COMPLEX NUMBERS

95. Number systems. If our number system consisted of zero and positive integers only, the solution of an equation such as $3x - 2 = 0$ would be impossible; for no number in the system considered satisfies this equation. We can extend the number system so as to include the class of numbers to which the solution belongs. These new numbers are the **rational fractions.**

While the solution of $3x - 2 = 0$ is possible in a number system composed of zero, positive integers, and rational fractions, the solution of an equation such as $x + 4 = 0$ is impossible. To meet the demands of such equations, we find it expedient again to extend the number system so as to include the negative numbers. In a number system thus extended an equation $ax + b = 0$, where a and b are any integers or fractions, has a solution.

The solution of an equation such as $x^2 = 2$ demands a further extension of our number system. It must be made to include irrational numbers, that is, numbers which cannot be represented by the quotient of two integers (see pp. 63, 64). But the number system thus extended is not sufficient to meet all the demands of the equations met in algebra. In this number system it is impossible to solve certain quadratic equations, for example, the equations $x^2 + 1 = 0$ and $x^2 - 6x + 13 = 0$. It is necessary again to extend the system so as to include numbers of the form $a + bi$, where a and b are real numbers, discussed in Art. 1; and where i is a symbol whose square is -1, that is, $i = \sqrt{-1}$ (see Art. 47). These numbers are usually called **complex numbers** and sometimes **imaginary numbers.** When $a = 0$ they are called **pure imaginary numbers.**

The term "imaginary number" is here used in a technical sense. The numbers are imaginary in the same sense that a fraction, a negative number, or an irrational number is imaginary for a number system consisting of positive integers.

At this point the question may be asked, — In working with this new system which includes complex numbers, may we not find it necessary to add new numbers, at present unknown, just as we found it necessary to add fractions, negative numbers, and irrational numbers to our system of positive integers? The answer to this question is that the system of complex numbers is sufficient to meet the demands of the equation.

While we have seen that the solution of equations with integral coefficients demands fractions, negative numbers, irrational numbers, and complex numbers, it is not to be inferred that all numbers are roots of equations with integral or rational coefficients. For example, the irrational number π cannot be the root of an equation with rational coefficients. The proof is beyond the scope of this book.*

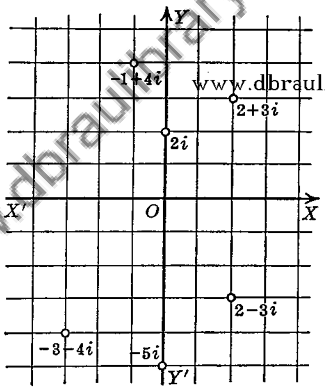

FIG. 29

96. Graphical representation of complex numbers. We have seen that all real numbers may be represented by points on a straight line. The complex number $x + iy$ depends on two real numbers x and y, and may be represented graphically by a point in a plane. Two lines, $X'X$, $Y'Y$, are drawn perpendicular to each other and intersecting at O, Fig. 29. To represent the number $2 + 3i$, measure off on $X'X$ to the right the distance 2, and up the distance 3. In general, the graph of the number $x + iy$ is the point whose coördinates are (x, y). The line $X'X$ is often called the axis of "reals" and the line $Y'Y$ the axis of imaginaries.

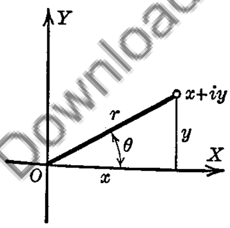

FIG. 30

★ It is often convenient to represent complex numbers by another method. Connect the point which represents $x + iy$ with the origin as in Fig. 30. Let the length of this line be r. The point can then be represented by giving

* See Klein, *Famous Problems in Elementary Geometry*, translation by Beman and Smith, p. 68.

★ The remainder of this article and the articles marked ★ may be omitted by those who have not studied trigonometry.

the length r and the angle θ. From the figure, using the definitions of sine and cosine given in trigonometry, we have

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$x^2 + y^2 = r^2.$$

Hence, the number $x + iy$ may be written in the form

$$x + iy = r(\cos\theta + i\sin\theta).$$

This form is called the **polar** form of a complex number. The angle θ is called the **argument or amplitude,** the length r the **modulus** or **absolute value** of the complex number. It should be noted that the complex numbers include all real numbers. In Fig. 29, the real numbers are represented by points on the line $X'X$. The pure imaginary numbers are represented by points on the line $Y'Y$.

97. Equal complex numbers. If two complex numbers $a + bi$ and $c + di$ are equal, then $a = c$ and $b = d$. For, if

$$a + bi = c + di, \tag{1}$$

by transposing,
$$a - c = (d - b)i. \tag{2}$$

Unless $a - c = d - b = 0$, we should have $a - c$, a real number, equal to $(d - b)i$, an imaginary number.

Conversely, if, $a = c$, and $b = d$,

$$a + bi = c + di.$$

Hence, *when any two expressions containing imaginary and real terms are equal to each other, we may equate the real parts and the imaginary parts separately.*

In particular, if $a + bi = 0$, $a = 0$ and $b = 0$.

EXERCISES

Represent graphically the following numbers and in each case find the argument and the modulus:

1. $2 - 3i$.

Solution: The number is represented in Fig. 29.

The modulus r is given by

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 9} = \sqrt{13}.$$

To find the argument θ we have

$$\tan\theta = \frac{y}{x} = -\frac{3}{2}, \ \theta = \text{arc tan} -\frac{3}{2}; \ \sin\theta = -\frac{3}{\sqrt{13}}.$$

2. $2 + 3i$

3. $-4 + 3i.$

4. $1 + i.$

5. $1 - 3i.$

6. $6 - 8i.$

7. $-i.$

8. $7i.$

9. $-7i.$

10. $\frac{7i}{3}.$

11. $6 + 0i.$

12. $-7.$

13. $\frac{3}{4} - \frac{i}{2}.$

14. $0.6 + 1.2i$

Write the following complex numbers in the form $x + iy$:

15. $6(\cos 30° + i \sin 30°).$

Solution: We have $r = 6, \theta = 30°.$

$$x = r \cos \theta = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}.$$

$$y = r \sin \theta = 6 \cdot \frac{1}{2} = 3.$$

Hence, $\quad 6(\cos 30° + i \sin 30°) = 3\sqrt{3} + 3i.$

16. $4(\cos 60° + i \sin 60°).$

17. $2(\cos 120° + i \sin 120°).$

18. $6(\cos 90° + i \sin 90°).$

19. $\cos 135° + i \sin 135°.$

20. $8(\cos 0° + i \sin 0°).$

21. $\frac{5}{2}(\cos 270° + i \sin 270°).$

22. If $(x + 2) + i(y - 2) = 0$, what are the values of x and y?

What must be the value of x and y in order that the following equations may be true?

23. $x - y + i(x + y) = 2 + 6i.$

24. $2x + 7y + i(3x - 2y) = 3 - 13i.$

25. $2x^2 + y^2 + i(x - y) = 1 + i.$

26. $x + 2y + xyi + xi - 3i - 5 = 0.$

98. Addition and subtraction of complex numbers. We assume that the number i like other numbers obeys all the laws of algebra. Given two complex numbers $a + bi$, $c + di$, we may write the sum and difference:

Thus, $\quad (a + bi) + (c + di) = (a + c) + (b + d)i,$

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

Hence, to add (or subtract) complex numbers, add (or subtract) the real and imaginary parts separately. The result is a complex number.

To add two complex numbers, $a + bi$ and $c + di$, graphically, we represent the numbers as points A and B in Fig. 31. Connect each point with the origin O. Complete

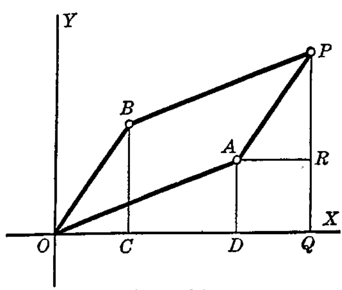

FIG. 31

the parallelogram, having OA and OB for adjacent sides. The vertex P represents the sum of the two given complex numbers. For, from the figure the coördinates of the fourth vertex are OQ and QP. But

$$OQ = OD + DQ = a + c,$$
$$QP = QR + RP = b + d.$$

Hence, P represents the point $(a + c) + (b + d)i$ which is the sum of $a + bi$ and $c + di$.

To subtract one complex number from another graphically, say $c + di$ from $a + bi$, we graph the points which represent $-c - di$ and $a + bi$, and proceed as for addition.

EXERCISES

Perform the following operations algebraically and graphically:

1. $(7 + 2i) + (1 + i)$.
2. $(2 + i) + (-3 + 2i)$
3. $(3 + 4i) + (1 - 5i)$.
4. $(-3 - i) + (2 + 2i)$.
5. $(1 + 2i) - \left(\frac{1}{2} + \frac{3}{2}i\right)$.
6. $(-2 + i) - (-2 - i)$.
7. $(4 + 7i) + (4 - 7i)$.
8. $(0 + 2i) + (2 - 5i)$.
9. $(1 + 2i) + (3 - 4i) - (5 - 6i)$.
10. $(5 + 0i) + (-2 - 4i) - (3 - 4i)$.

★ **99. Multiplication of complex numbers.** Let $a + ib$ and $c + id$ be any two complex numbers. Since i obeys all the laws of algebra, we have

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad).$$

The result is a complex number. To multiply two complex numbers graphically, let the two numbers $a + ib$, $c + id$ be represented by the points P_1 and P_2 (Fig. 32). Reducing to the polar form, we have

$$a + bi = r_1(\cos \theta_1 + i \sin \theta_1),$$
$$c + di = r_2(\cos \theta_2 + i \sin \theta_2).$$

By actual multiplication,

$$(a + bi)(c + di)$$
$$= r_1r_2[\cos \theta_1 \cos \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) - \sin \theta_1 \sin \theta_2]$$
$$= r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].$$

Hence, *the modulus of the product of two complex numbers is the product of their moduli and the argument is the sum of their arguments.*

The point P, which represents $(a + bi)(c + di)$ may then be constructed by drawing through O a line making an angle $\theta = \theta_1 + \theta_2$ with the line OX (Fig. 32) and constructing on this line a segment OP whose length is r_1r_2.

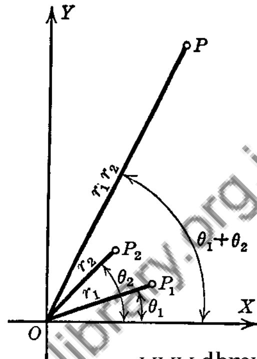

FIG. 32

100. Conjugate complex numbers. Numbers which differ only in the sign of the imaginary parts are called **conjugate numbers.** Thus, $3 + 2i$ and $3 - 2i$ are conjugate. Since

$$(a + bi) + (a - bi) = 2a,$$

$$(a + bi)(a - bi) = a^2 + b^2,$$

and $$(a + bi) - (a - bi) = 2bi,$$

we see that the sum and the product of two conjugate complex numbers are real numbers, but the difference of two conjugate complex numbers is an imaginary number.

EXERCISES

Multiply both analytically and graphically, finding the arguments and moduli of the products.

1. $(3 + \sqrt{3}i)(2 + 2i)$.

Solution:

$$(3 + \sqrt{3}i)(2 + 2i) = 6 + 6i + 2\sqrt{3}i + 2\sqrt{3}i^2 = 6 - 2\sqrt{3} + i(6 + 2\sqrt{3}).$$

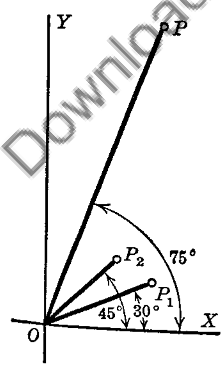

FIG. 33

Putting the numbers in the polar form, we have,

$$3 + \sqrt{3}i = 2\sqrt{3}(\cos 30° + i \sin 30°),$$

$$2 + 2i = 2\sqrt{2}(\cos 45° + i \sin 45°).$$

Hence,

$$r_1 = 2\sqrt{3},\ r_2 = 2\sqrt{2},\ \theta_1 = 30°,\ \theta_2 = 45°.$$

The modulus of the product is, then,

$$r_1r_2 = 4\sqrt{6},$$

and the argument is 75°.

Let P_1 and P_2 in Fig. 33 represent the two given numbers. Through O draw a line making an angle of 75° with the line OX. On this line measure off the distance

$$OP = 4\sqrt{6}.$$

The point P then represents the product of the two numbers.

2. $(2 + 2i)(3 + 3i)$.

3. $(3 + i\sqrt{3})(3 + i\sqrt{3})$.

4. $(2 - 2i)(3 + 3i)$.

5. $(2 - 2i)(-2 - 2i)$.

6. $(1 - i)^2$.

7. $(1 - i)^3$.

8. $(1 - i)^4$.

9. $(1 - i)(1 - 2i)(1 - 3i)$.

10. $(0 + 7i)(3 - 3i)$.

11. $(3 + 0i)(1 - \sqrt{3}i)$.

12. $(0 + 4i)^2(0 - 4i)^2$.

★ **101. De Moivre's theorem.** If two complex numbers are equal, then as a special case of Art. 99, we have

$$\begin{aligned} r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) &= r^2(\cos\theta + i\sin\theta)^2 \\ &= r^2(\cos 2\theta + i\sin 2\theta). \end{aligned}$$

Multiplying both sides of this identity by $r(\cos\theta + i\sin\theta)$, we have

$$r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta),$$

and it can easily be proved by mathematical induction that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta),$$

where n is any positive integer.

This relation is known as **De Moivre's theorem,** and holds also for fractional values of the exponent.

To prove the theorem when the exponent is the reciprocal of a positive integer, consider the expression $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$ in which n is a positive integer.

Let $$\theta = n\phi,$$

then

$$\begin{aligned} (\cos\theta + i\sin\theta)^{\frac{1}{n}} &= (\cos n\phi + i\sin n\phi)^{\frac{1}{n}} \\ &= [(\cos\phi + i\sin\phi)^n]^{\frac{1}{n}} = \cos\phi + i\sin\phi \\ &= \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}. \end{aligned}$$

De Moivre's theorem thus gives an easy method of finding any power or any root of a complex number.

The proof can be readily extended to the case of an exponent which is any rational fraction.

★ **102. Roots of complex numbers.** From Art. 101, the nth root of $x + iy$ is

$$(x + iy)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).$$

If m be any integer, $\cos(\theta + m\,360°) = \cos\theta,$
$\sin(\theta + m\,360°) = \sin\theta.$

We may then write

$$(x + iy)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = [r\{\cos(\theta + m\,360°) + i\sin(\theta + m\,360°)\}]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\frac{\theta + m\,360°}{n} + i\sin\frac{\theta + m\,360°}{n}\right].$$

If now we let m take the values $0, 1, 2, 3, \cdots, n - 1$, we find n results, all different numbers whose nth powers are $x + iy$. We may then state the following

THEOREM. *Any number has n distinct nth roots.*

EXERCISES

Using De Moivre's theorem, find the indicated powers and roots.

1. $(3 + \sqrt{3}i)^4$.

Solution: Writing $3 + \sqrt{3}i$ in the polar form,

$$3 + \sqrt{3}i = 2\sqrt{3}(\cos 30° + i\sin 30°).$$

By De Moivre's theorem,

$$\begin{aligned}(3 + \sqrt{3}i)^4 &= [2\sqrt{3}(\cos 30° + i\sin 30°)]^4\\ &= 144(\cos 120° + i\sin 120°)\\ &= 144\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i\right)\\ &= -72 + 72\sqrt{3}i.\end{aligned}$$

2. $(3 + \sqrt{3}i)^2$.

3. $(2 + 2i)^3$.

4. $(3 + \sqrt{3}i)^5$.

5. $\left(\frac{1}{2} - \frac{1}{2}\sqrt{3}i\right)^6$.

6. $[2 \cdot (\cos 30° + i\sin 30°)]^{10}$.

7. $(1 + i)^8$.

8. $(1 - i)^8$.

9. $\sqrt[3]{-2 + 2i}$.

Solution: Writing $-2 + 2i$ in the polar form, we have

$$-2 + 2i = 2\sqrt{2}(\cos 135° + i\sin 135°).$$

By De Moivre's theorem,

$$\begin{aligned}\sqrt[3]{-2+2i} = (-2 + 2i)^{\frac{1}{3}} &= [2\sqrt{2}\{\cos(135° + m360°) + i\sin(135° + m360°)\}]^{\frac{1}{3}}\\ &= \sqrt{2}[\cos(45° + m\,120°)1 + i\sin(45° + m\,120°)].\end{aligned}$$

For $m = 0, 1$, and 2, this expression reduces to

$$1 + i,\ \sqrt{2}(\cos 165° + i\sin 165°),$$

and

$$\sqrt{2}(\cos 285° + i\sin 285°)$$

respectively. Any one of these three numbers is a cube root of $-2+2i$. The points P_1, P_2, P_3, representing these three numbers lie at equal intervals on a circle of radius $\sqrt{2}$ (Fig. 34).

10. $\sqrt[3]{2-2i}$.

11. $\sqrt{2+2\sqrt{3}i}$.

12. $\sqrt{(\cos 60° + i \sin 60°)}$.

13. $\sqrt[3]{27(\cos 60° + i \sin 60°)}$.

14. $\sqrt[3]{\cos 360° + i \sin 360°}$.

15. $\sqrt[3]{\cos 21° + i \sin 21°}$.

16. $\sqrt[3]{1}$.

Hint: Write in the form $\sqrt[3]{1} = \sqrt[3]{\cos 0° + i \sin 0°}$.

17. $\sqrt[3]{i}$.

Hint: Write in the form $\sqrt[3]{\cos 90° + i \sin 90°}$.

18. $\sqrt[3]{27}$. **19.** $\sqrt[4]{1}$. **20.** $\sqrt{5+12i}$.

21. If ω represents one of the complex cube roots of unity found in exercise 16 and ω_1 the other, verify that $\omega_1 = \omega^2$, and that $\omega = \omega_1^2$.

Find all the roots of the following equations and represent them graphically:

22. $x^3 - 1 = 0$.

Hint: $x^3 = 1$. The roots of the equation are then the three cube roots of unity. See exercise 16.

23. $x^3 - 8 = 0$.

24. $x^5 - 32 = 0$.

25. $x^5 - 1 = 0$.

26. $x^4 - 16 = 0$.

27. $x^6 - 1 = 0$.

28. $x^8 - 1 = 0$.

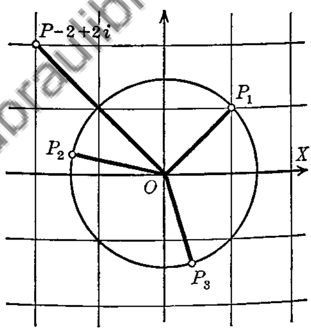

FIG. 34

★ 103. Division of complex numbers. The quotient of two complex numbers may be obtained as follows:

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\cdot\frac{c-id}{c-id} = \frac{ac+bd-i(ad-bc)}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} - i\,\frac{ad-bc}{c^2+d^2}.$$

This is a complex number. Writing the two given complex numbers in the polar form, we have

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

Hence, *the modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument is the difference of their arguments.*

If, in Fig. 35, P_1 and P_2 represent the points $a + ib$ and $c + id$ respectively, the point P which represents the quotient $\frac{a + ib}{c + id}$ may be constructed by drawing through O a line making an angle $\theta = \theta_1 - \theta_2$ with the line OX, and constructing on this line a segment OP, whose length is $\frac{r_1}{r_2}$.

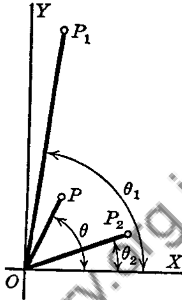

FIG. 35

EXERCISES

Find the quotients of the following pairs of numbers, analytically and graphically.

1. $(4 + 4i) \div \left(2 + \frac{2}{3}\sqrt{3}i\right)$.

Solution: $\frac{4 + 4i}{2 + \frac{2}{3}\sqrt{3}i} = \frac{4 + 4i}{2 + \frac{2}{3}\sqrt{3}i} \cdot \frac{2 - \frac{2}{3}\sqrt{3}i}{2 - \frac{2}{3}\sqrt{3}i} = \frac{3 + \sqrt{3}}{2} + \frac{3 - \sqrt{3}}{2}i.$

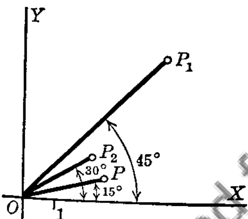

FIG. 36

Writing both numbers in the polar form, we obtain

$$4 + 4i = 4\sqrt{2}(\cos 45° + i \sin 45°),$$

$$2 + \frac{2}{3}\sqrt{3}i = \frac{4}{3}\sqrt{3}(\cos 30° + i \sin 30°).$$

Hence, $r_1 = 4\sqrt{2}$, $r_2 = \frac{4}{3}\sqrt{3}$, $\theta_1 = 45°$, $\theta_2 = 30°$.

The modulus of the quotient is then $\frac{r_1}{r^2} = \sqrt{6}$, and the argument is $\theta = \theta_1 - \theta_2 = 15°$. Let P_1 and P_2 in Fig. 36 represent the two given numbers. Through O draw a line, making an angle of 15° with the line OX. Measure off on this line the distance $OP = \sqrt{6}$. The point P then represents the quotient of the two numbers.

2. $(1 - i) \div (1 + i)$.
3. $(1 + i) \div (2 - 2i)$
4. $(2 + 2i) \div (3 + \sqrt{3}i)$.
5. $4 \div (1 + i)$.
6. $4i \div (1 + i)$.
7. $(1 + i) \div 4i$.
8. $\left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right) \div \left(-\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)$.
9. $(2 - i) \div (3 + i)$.
10. $13i \div (2 + 3i)$.

Find the reciprocals of the following complex numbers. Give a graphical representation of each number and its reciprocal.

11. $1 - i$.
12. $1 + i$.
13. $2 + 3i$.
14. i.
15. $1 - 2i$.
16. $\frac{1}{4} + \frac{1}{4}i$.

CHAPTER XIV

THEORY OF EQUATIONS

104. General equation of degree *n*. In the present chapter, we shall study rational integral equations in one unknown. As stated in Art. 25, such an equation of degree n in x, can be written in the form

$$a_0x^n + a_1x^{n-1} \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

where n is a positive integer, $a_0, a_1, \cdots, a_{n-1}, a_n$ do not involve x, and $a_0 \neq 0$. The equation (1) is often called the **general equation of the *n*th degree in *x*.**

The function

$$f(x) = a_0x^n + a_1x^{n-1} \cdots + a_{n-1}x + a_n$$

is the polynomial of degree n in x (see Art. 24). The polynomial of the second degree has been discussed in connection with the quadratic equation.

When $f(x)$ is used in this chapter, it is to be understood to mean a polynomial in x.

ORAL EXERCISES

By comparing the following polynomials with the general form, determine $n, a_0, a_1, \cdots, a_n$.

(a) $$f(x) = \frac{3}{5}x^5 + \frac{2}{7}x^4 + 8.$$

(b) $$f(x) = \frac{4}{7}x^3 + \frac{2}{5}x^4 + 10x^2 + ix^2.$$

(c) $$f(x) = (i + \sqrt{3})x^5 + 5x^2 + 10.$$

The principal object of this chapter is to present methods which aid in determining exactly or approximately the real roots of special numerical * equations included under type (1). It is largely for this purpose that we discuss the graphs of polynomials. The **zeros** of the polynomial are the **roots** of the equation formed by equating the polynomial to zero. The real roots of the equation may then be looked upon geometrically as the abscissas of

* The term "numerical equations" is here used to indicate that the coefficients are not literal.

the points of the X-axis where the graph of the polynomial meets this axis. The remainder theorem (Art. 105) and synthetic division (Arts. 108, 109) often save much time in carrying out the graphic representation of polynomials.

105. Remainder theorem. *If r is some constant and if $f(x)$ is divided by $(x - r)$ until a remainder is obtained that does not involve x, this remainder is equal to $f(r)$.*

If a polynomial $f(x)$ be divided by $(x - r)$, denote the quotient by $Q(x)$ and the remainder described in the theorem by R. Then

$$f(x) \equiv (x - r)Q(x) + R. \tag{1}$$

If we put $x = r$ in this identity, we obtain

$$f(r) = (r - r)Q(r) + R.$$

But since $(r - r)Q(r) = 0$, it follows that

$$R = f(r). \tag{2}$$

Example. Without performing the division, find the remainder when $x^3 + 5x^2 + 3x - 1$ is divided by $(x + 2)$.

Solution. Here $f(x) = x^3 + 5x^2 + 3x - 1$,

$x - r = x + 2$, thus $r = -2$.

Then $f(r) = (-2)^3 + 5(-2)^2 + 3(-2) - 1 = 5.$

Hence, the remainder is 5.

Corollary. *If $f(r) = 0$, then $f(x)$ is exactly divisible by $(x - r)$.*

This corollary follows at once from (1) and (2).

Example. Without performing the division, show by the corollary that $x^5 + 32$ is exactly divisible by $(x + 2)$.

106. Factor theorem. *If r is a root of the equation $f(x) = 0$, then $(x - r)$ is a factor of $f(x)$.*

Since the hypothesis that r is a root of the equation $f(x) = 0$ means that $f(r) = 0$, this theorem follows directly from the corollary to the remainder theorem (Art. 105).

107. Converse of the factor theorem. *If $(x - r)$ is a factor of $f(x)$, then r is a root of the equation $f(x) = 0$.*

By hypothesis, $f(x)$ is exactly divisible by $(x - r)$. Thus, in the notation of Art. 105,

$$f(x) = (x - r)Q(x).$$

Hence, $f(r) = 0 \cdot Q(r) = 0$, which means that r is a root of the equation $f(x) = 0$.

EXERCISES

1. Find the remainder when $x^2 - 7x + 14$ is divided by $(x - 3)$:

(*a*) by performing the division;

(*b*) by the use of the remainder theorem.

Without performing the divisions, find the remainder after each of the following divisions, by the use of the remainder theorem:

2. $(3x^4 - 4x^3 + 5x^2 - 5x + 1) \div (x - 1)$.

3. $(3x^4 - 4x^3 + 5x^2 - 5x + 1) \div (x + 1)$.

4. $(x^3 - 5x^2 + 3x - 1) \div (x - 3)$.

5. $(x^3 - 5x^2 + x - 3) \div x$.

6. $(x^3 - 2x^2 + 4x - 1) \div (x + 2)$.

7. Show that $(x - 2)$ is a factor of $x^3 - 7x + 6$, by the use of the corollary to the remainder theorem, and also by actual division.

8. Given $f(x) = x^5 + a^5$. Find $f(-a)$. Is $x + a$ a factor of $x^5 + a^5$?

9. Given that 2 is a root of the equation $x^4 - 16 = 0$, write a linear factor of $x^4 - 16$. Find also a third degree factor of $x^4 - 16$.

10. Show, by the remainder theorem, that $x^n - a^n$ is exactly divisible by $(x + a)$ if n is even.

11. Show that $x^n + a^n$ is divisible by $(x + a)$ if n is odd.

12. By means of the remainder theorem, find a value for k such that $x^3 + 3kx^2 - 4x + 10$ is divisible by $x + 2$.

108. Synthetic division. Since we shall often have occasion to divide a polynomial, $f(x)$, by (x − an assigned number), (1) in finding values of $f(x)$ for assigned values of x, (2) in finding factors of $f(x)$, and (3) in solving the equation $f(x) = 0$, it is important to learn a short method of performing the divisions. We shall now illustrate and develop Horner's **method of synthetic division** for dividing $f(x)$ by $(x - r)$.

Illustration: Divide $5x^4 - 6x^3 + 8x^2 - 24x - 6$ by $(x - 2)$.

By the ordinary method

$$
\begin{array}{l|l}
5x^4 - 6x^3 + 8x^2 - 24x - 6 & x - 2 \\ \cline{2-2}
\underline{5x^4 - 10x^3} & 5x^3 + 4x^2 + 16x + 8 \\
\qquad 4x^3 + 8x^2 & \\
\qquad \underline{4x^3 - 8x^2} & \\
\qquad\qquad 16x^2 - 24x & \\
\qquad\qquad \underline{16x^2 - 32x} & \\
\qquad\qquad\qquad 8x - 6 & \\
\qquad\qquad\qquad \underline{8x - 16} & \\
\qquad\qquad\qquad\quad + 10 &
\end{array}
$$

Manifestly, the work can be abridged by writing only the coefficients, thus,

$$\begin{array}{rrrrrl}
5 & -\;6 & +\;8 & -\;24 & -\;6 & \big|\,\underline{1-2\qquad\qquad} \\
5 & -\;10 & & & & \;5+4+16+8 \\ \cline{1-2}
 & +\;4 & +\;8 & & & \\
 & +\;4 & -\;8 & & & \\ \cline{2-3}
 & & +\;16 & -\;24 & & \\
 & & +\;16 & -\;32 & & \\ \cline{3-4}
 & & & +\;8 & -\;6 & \\
 & & & +\;8 & -\;16 & \\ \cline{4-5}
 & & & & +\;10 &
\end{array}$$

Since the coefficient of x in $x - r$ is unity, the coefficient of the first term of each remainder is the coefficient of the next term to be obtained in the quotient. Further, it is not necessary to write the terms of the dividend as part of the remainder, nor the first term of the partial products.

The work thus becomes:

$$\begin{array}{rrrrrl}
5 & -\;6 & +\;8 & -\;24 & -\;6 & \big|\,\underline{1-2} \\
 & -\;10 & & & & \\ \cline{1-2}
 & +\;4 & & & & \\
 & & -\;8 & & & \\ \cline{1-3}
 & & +\;16 & & & \\
 & & & -\;32 & & \\ \cline{3-4}
 & & & +\;8 & & \\
 & & & & -\;16 & \\ \cline{4-5}
 & & & & +\;10 &
\end{array}$$

We may omit the first term of the divisor and write the work in the following more compact form:

$$\begin{array}{rrrrrl}
5 & -\;6 & +\;8 & -\;24 & -\;6 & \big|\,\underline{-2} \\
 & -\;10 & -\;8 & -\;32 & -\;16 & \\ \hline
5 & +\;4 & +\;16 & +\;8 & +\;10 &
\end{array}$$

If we replace -2 by $+2$, we may add the partial products to the numbers in the dividend. Then, we have:

$$\begin{array}{rrrrrl}
5 & -\;6 & +\;8 & -\;24 & -\;6 & \big|\,\underline{2} \\
 & +\;10 & +\;8 & +\;32 & +\;16 & \\ \hline
5 & +\;4 & +\;16 & +\;8 & +\;10 &
\end{array}$$

The quotient is $5x^3 + 4x^2 + 16x + 8$, and the remainder is 10.

109. Rule for synthetic division. *To divide $f(x)$ by $(x - r)$, arrange $f(x)$ in descending powers of x, supplying all missing powers by putting in zeros as coefficients.*

Detach the coefficients, write them in a horizontal line and in the order $a_0, a_1, a_2, \cdots, a_n$.

Bring down the first coefficient a_0; multiply a_0 by r, and add the product to a_1; multiply this sum by r, and add the product to a_2. Continue this process; the last sum is the remainder, and the pre-

ceding sums are the coefficients of the powers of x in the quotient, arranged in descending order.

Proof of Rule. This rule may be established by mathematical induction. By long division,

$$\begin{array}{l|l} a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_sx^{n-s}+a_{s+1}x^{n-s-1}+\cdots+a_n & x-r \\ \hline & a_0x^{n-1}+(a_1+a_0r)x^{n-2}+\cdots+(a_{s-1}+ra_{s-2}+\cdots+r^{s-1}a_0)x^{n-s} \end{array}.$$

$$\begin{array}{l} a_0x^n - a_0rx^{n-1} \\ \hline (a_1+a_0r)x^{n-1}+a_2x^{n-2} \\ (a_1+a_0r)x^{n-1}-(a_1r+a_0r^2)x^{n-2} \\ \hline \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{array}$$

We note that the coefficient of x^{n-2} in the quotient is formed according to the rule. Assume that the coefficients in the quotient down to that of x^{n-s} are formed according to the rule. On this hypothesis, proceed by long division to find the coefficient of x^{n-s-1} in the quotient. This may be exhibited as a continuation of the division above as follows:

$$\begin{array}{l} (a_{s-1}+ra_{s-2}+\cdots+r^{s-1}a_0)x^{n-s+1}+a_sx^{n-s}+a_{s+1}x^{n-s-1}+\cdots+a_n \\ (a_{s-1}+ra_{s-2}+\cdots+r^{s-1}a_0)x^{n-s+1}-(ra_{s-1}+r^2a_{s-2}+\cdots+r^sa_0)x^{n-s}. \\ \hline (a_s+ra_{s-1}+r^2a_{s-2}+\cdots+r^sa_0)x^{n-s}+a_{s+1}x^{n-s-1}+\cdots+a_n. \end{array}$$

This shows that if the coefficients in the quotient down to that of x^{n-s} are formed according to the rule, the coefficient of the next lower power is formed according to the rule. Hence, the rule is established.

EXERCISES

Divide by synthetic division and check by ordinary division.

1. x^4+3x^3-5x+3 by $(x-4)$.

Solution:

$$\begin{array}{rrrrr|l} 1+ & 3+ & 0- & 5+ & 3 & \underline{4} \\ & +4+ & 28+ & 112+ & 428 & \\ \hline 1+ & 7+ & 28+ & 107+ & 431 & \end{array}$$

The quotient is $x^3+7x^2+28x+107$ and the remainder is 431.

Perform the following divisions both by long division and by synthetic division:

2. $(x^3-4x^2+6x-3)\div(x-1)$.

3. $(x^3+4x^2-3x-6)\div(x+1)$. *Hint:* In this case, $r=-1$.

4. $(2x^3-3x^2+3)\div(x-2)$.

By synthetic division, find the quotient and remainder in the following:

5. $(3x^3-4x^2-5)\div(x-2)$.

6. $(x^3-27)\div(x-3)$.

7. $(x^4-x-3)\div(x+4)$.

8. $(7x^4-3x^2-2)\div(x-\frac{1}{2})$.

9. Given $f(x) = 3x^3 - 5x^2 - x + 10$. Use the remainder theorem and synthetic division to find each of the following: $f(2), f(-1), f(3)$.

10. Given $f(x) = 6x^3 + x^2 - 31x + 24$. Find $f(-2), f(-1), f(1), f(2)$.

110. Graphs of polynomials. When the coefficients of $f(x)$ are real numbers, the march of the function for different values of x can be clearly presented by the use of the graphic methods explained in Arts. 15, 16. To any assigned value of x, there corresponds one and only one value of the polynomial $f(x)$. This is sometimes expressed by saying that $f(x)$ is single valued. The fact that the graph of $f(x)$ is a continuous curve (see Art. 16) makes it of much service in the theory of equations.

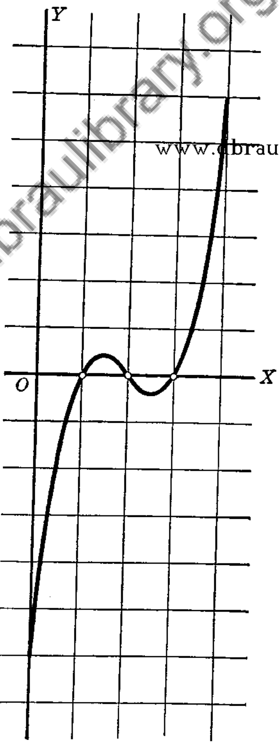

FIG. 37

EXERCISES

Construct the graphs of the following functions and locate their real zeros approximately (to within 0.5):

1. $f(x) = x^3 - 6x^2 + 11x - 6$.

As pointed out in Art. 108, synthetic division furnishes a convenient method of evaluating $f(x)$ for different values of x. Thus $f(0.5)$ is obtained as follows:

$$\begin{array}{rrrr|l} 1 & -6 & +11 & -6 & \underline{0.5} \\ & 0.5 & -2.75 & +4.125 & \\ \hline 1 & -5.5 & +8.25 & -1.875 & \end{array}$$

Hence, $f(0.5) = -1.875$.

In this way the following values are obtained:

$f(-2) = -60$.
$f(-1) = -24$.
$f(-0.5) = -13.125$.
$f(0) = -6$.
$f(0.5) = -1.875$.
$f(1) = 0$.
$f(1.5) = 0.375$.
$f(2) = 0$.
$f(2.5) = -0.375$.
$f(3) = 0$.
$f(4) = 6$.
$f(5) = 24$.

The graph is shown in Fig. 37; it presents to the eye the following facts:

(1) $f(x)$ has zeros at 1, 2, and 3.
(2) $f(x)$ is positive when $2 > x > 1$, and when $x > 3$.
(3) $f(x)$ is negative when $x < 1$ and when $3 > x > 2$.

Query. If x were assigned a numerically very large value, say $x = 100$ or $x = -100$, how would the value of the highest degree term, x^3, compare in numerical value with all the other terms together?

2. $x^3 - 6x^2 + 8x$.
3. $x^3 - 12x + 3$.
4. $x^3 - x^2 - 7x + 3$.
5. $x^4 - 2x^3 - 7x^2 + 8x + 12$.
6. $2x^4 - 3x^3 - 20x^2 + 27x + 18$.
7. $3x^4 - 4x^3 - 12x^2 + 3$.
8. $x^3 - 2x - 4$.

111. Graphical solution of an equation $f(x) = 0$. The real roots of an equation $f(x) = 0$ are the abscissas of the points where $f(x)$ meets the X-axis. It is then obvious that in locating the zeros of certain functions, $f(x)$, in Art. 110, we are also locating the roots of $f(x) = 0$.

ORAL EXERCISES

Give the roots of each of the following equations:

1. $x^3 - 6x^2 + 11x - 6 = 0$. (See exercise 1, Art. 110.)
2. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$. (See exercise 5, Art. 110.)

112. Number of roots of an equation. *Every equation, $f(x) = 0$, of the nth degree has n roots and no more.*

To prove this proposition we assume the fundamental theorem that every equation, $f(x) = 0$, has at least one root. More explicitly, we assume that

*There always exists at least one number, real or complex, which will satisfy an equation of the nth degree, whose coefficients are any real or complex numbers.**

Let r_1 be a root of $f(x) = 0$, then (Art. 106) $(x - r_1)$ is a factor of $f(x)$ and

$$f(x) = 0 \text{ becomes } (x - r_1)f_1(x) = 0, \tag{1}$$

where $f_1(x)$ is a polynomial of degree $n - 1$, beginning with the term a_0x^{n-1}. By the theorem assumed, $f_1(x) = 0$ has at least one root. Let r_2 be a root; then

$$f_1(x) = 0 \text{ becomes } (x - r_2)f_2(x) = 0$$

and

$$f(x) = 0 \text{ becomes } (x - r_1)(x - r_2)f_2(x) = 0, \tag{2}$$

in which $f_2(x)$ is a polynomial of degree $n - 2$, beginning with the term a_0x^{n-2}. Continuing this process, we separate out n linear factors with a quotient a_0, so that

$$f(x) = 0 \text{ becomes } a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0, \tag{3}$$

where $r_1, r_2, \cdots, r_n$ are n roots of $f(x) = 0$.

* This fundamental theorem was first proved by Gauss in 1797. For proof, see Fine's *College Algebra*, p. 588.

If $f(x) = 0$ has another root different from any of these, let r denote such a root. Then, from (3),

$$a_0(r - r_1)(r - r_2) \cdots (r - r_n) = 0. \tag{4}$$

But here we should have the product of factors equal to zero when no one of the factors is zero. As this is impossible (III, Art. 5), there are not more than n roots of $f(x) = 0$. Hence, every equation of the nth degree has n roots and no more. Furthermore, every polynomial of the nth degree is the product of n linear factors. It is not, however, possible, in general, to determine these factors if n exceeds 4 (see Art. 128). Two or more of the n roots of $f(x) = 0$ may be equal to each other. Equal roots are called **multiple roots.** If the same root occurs twice, it is called a **double root;** if three times, a **triple root;** if m times ($m > 3$), a **root of multiplicity** $\boldsymbol{m}$. Thus, $(x - 2)^5 = 0$ has the root 2 of multiplicity 5, and $(x - 1)^2(x - 3)(x - 4)^3 = 0$ has a double root equal to 1, a single root equal to 3, and a triple root equal to 4.

COROLLARY I. *If two polynomials of degrees not greater than n are equal to each other for more than n distinct values of the variable, the coefficients of like powers of the variable are equal.*

Let

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n \tag{4}$$

for more than n values of x.

From (4),

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0. \tag{5}$$

Then

$$\begin{aligned} a_0 - b_0 &= 0, \\ a_1 - b_1 &= 0, \\ &\cdots \\ a_n - b_n &= 0. \end{aligned}$$

For, if any coefficient in (5) were not equal to zero, we should have an equation of degree equal to or less than n with more than n roots, which is contrary to the theorem just proved.

Hence, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

COROLLARY II. *If two polynomials of degrees not greater than n are equal for more than n distinct values of the variable, they are equal for all values, and the equality is an identity.*

113. Forming an equation with given roots. If $r_1, r_2, \cdots, r_n$ are given roots of an equation $f(x) = 0$, then it follows from Art. 112 that the equation may be written in the form

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

in which we may assign to a_0 any value not equal to 0.

In case all the roots are integers or rational fractions, a_0 may be selected as an appropriate multiple of the denominators of the fractions in such a manner that all the coefficients of the terms of the equation are integers.

Illustration 1. Form an equation that has roots $1, -1, \frac{1}{2}, \frac{2}{3}$ and no others.

Solution: The equation is of the form

$$a_0(x - 1)(x + 1)(x - \tfrac{1}{2})(x - \tfrac{2}{3}) = 0. \tag{1}$$

By choosing $a_0 = 6$, we obtain an equation free from fractional coefficients. This choice of a_0 is equivalent to multiplying the last factor of the left-hand side of (1) by 3, and the preceding factor by 2, thus obtaining

$$(x - 1)(x + 1)(2x - 1)(3x - 2) = 0,$$

or

$$6x^4 - 7x^3 - 4x^2 + 7x - 2 = 0.$$

Illustration 2. Form an equation with integral coefficients that has 1 as a triple root, $\frac{1}{2}$ as a double root, $\frac{1}{3}$ as a single root, and no other roots.

Solution: The equation is, following illustration 1,

$$(x - 1)^3(2x - 1)^2(3x - 1) = 0,$$

or

$$12x^6 - 52x^5 + 91x^4 - 82x^3 + 40x^2 - 10x + 1 = 0.$$

EXERCISES

Find each root of the equation and indicate the multiplicity of each root.

1. $5(x - 2)(x - 3)^2(x + 1) = 0.$
2. $(3x - 1)(2x + 1)^3(x + 2) = 0.$
3. $(x - 1)(x^2 + x + 1) = 0.$
4. $x^4 - 3x^3 + 2x^2 = 0.$
5. $(4x - 1)^5(x - 3) = 0.$
6. $(3x - 2)^6(x + 8) = 0.$
7. Show that 2 is a double root of $2x^3 - 9x^2 + 12x - 4 = 0.$
8. Show that 3 and $\frac{1}{3}$ are each double roots of

$$9x^5 - 51x^4 + 58x^3 + 58x^2 - 51x + 9 = 0,$$

and find the other root.

9. Form equations that have the following roots and no others.
   (a) 2, 3, 5.
   (b) $1 + \sqrt{2}, 1 - \sqrt{2}, 3.$
   (c) $1 + 2i, 1 - 2i$, when $i^2 = -1.$

10. Form equations with integral coefficients that have the following roots and no others.

(*a*) a double root 3, and single roots $\frac{1}{2}$ and $\frac{2}{3}$.

(*b*) a triple root -1, and a double root $\frac{1}{2}$.

11. By means of the theorem concerning the number of roots of $f(x) = 0$, show: (1) if $f(x) = 0$ be multiplied by a polynomial in x, the resulting equation has more roots than $f(x) = 0$; (2) if $f(x) = 0$ be divided by a polynomial in x, which is a factor of $f(x)$, the resulting equation will have fewer roots than $f(x) = 0$.

114. Comments on the graphs of factored polynomials.

Given

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

We assume, for the present, that $a_0, a_1, \cdots, a_n$ are real numbers, and further for convenience of expression that a_0 is positive, although this is not a necessary limitation. In Art. 110 the graphs of a few polynomials are plotted. Some important properties of these graphs appear when the polynomial is separated into linear factors. We cannot at this point make the treatment so complete as later, but we may well consider two important cases:

1. *When the factors* $x - r_1, x - r_2, \cdots, x - r_n$ *are all real and distinct.*

Arrange the factors so that $r_1 > r_2 > \cdots > r_{n-1} > r_n$. When $x > r_1$ all the factors are positive and the graph is above the X-axis. When $r_1 > x > r_2$, one factor is negative and the graph is below

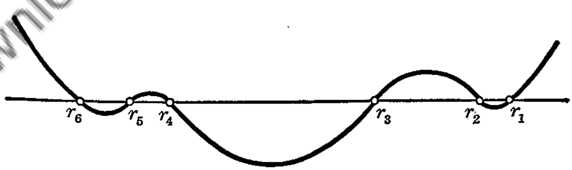

FIG. 38

the X-axis. When $r_2 > x > r_3$, two factors are negative, and the graph is again above the X-axis. Continuing this process, we see that the graph crosses the X-axis at the n points, $x = r_1$, $x = r_2$, $\cdots$, $x = r_n$, and we obtain a general notion of the nature of the curve. See Fig. 38.

2. *When the factors are real but some of them repeated.*

To discuss the graph in this case, take for example,

$$f(x) = a_0(x - r_1)^2(x - r_2)(x - r_3)^5,$$

and let

$$r_1 > r_2 > r_3.$$

Since the factors $x - r_2$ and $x - r_3$ occur to powers with odd exponents, it follows as above that the curve crosses the X-axis at $x = r_2$ and $x = r_3$. But it does not cross at $x = r_1$, since the sign of $f(x)$ is the same when $x > r_1$ as when $r_1 > x > r_2$, and the curve touches the axis at $x = r_1$. In general, *if a factor* $(x - r)^m$ *occurs where* m *is odd, the graph crosses the* x*-axis at* $x = r$ (see A and B, Fig. 39). *Moreover, if* $m = 1$, *the graph tends to cut the* x*-axis as at* B, Fig. 39, *whereas if* m *is an odd number greater than* 1, *the graph not only crosses at* $x = r$, *but is tangent to the* x*-axis at* $x = r$ *as at* A, Fig. 39. *If* h *is even, the graph merely touches the* x*-axis without crossing* (see C, Fig. 39). These comments, in italics, follow in part from studies in the calculus, and may be accepted here without proof.

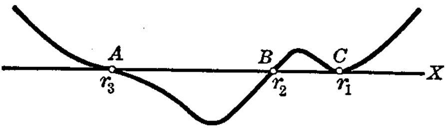

FIG. 39

Another case is discussed in Art. 116, where imaginary factors occur.

115. Theorem concerning imaginary roots. *If a complex number* $a + bi$ *is a root of an equation* $f(x) = 0$ *with real coefficients, the conjugate complex number* $a - bi$ *is also a root. Thus, imaginary roots occur in conjugate pairs.*

The theorem is established if we can show that $[x - (a - bi)]$ is a factor of $f(x)$ (see Art. 107). Since $[x - (a - bi)]$ is a factor of the quadratic expression

$$D(x) = x^2 - 2ax + a^2 + b^2 = [x - (a + bi)][x - (a - bi)], \quad (1)$$

our theorem is proved if we can show that $D(x)$ is a factor of $f(x)$. Divide $f(x)$ by the second degree expression, $D(x)$, until we obtain a first degree remainder, $cx + d$, and a quotient, $Q(x)$. Then we may write

$$f(x) = D(x)Q(x) + cx + d, \quad (2)$$

where c and d are real numbers since the quadratic function $D(x)$ has real coefficients.

Since, by hypothesis, $(a + bi)$ is a root of $f(x) = 0$, we have $f(a + bi) = 0$. From (1),

$$D(a + bi) = 0.$$

Hence, if we substitute $x = (a + bi)$ in (2), we get

$$0 = 0 \cdot Q(a + bi) + c(a + bi) + d,$$

or

$$(ac + d) + bci = 0. \tag{3}$$

Equating reals and imaginaries on the two sides of (3), we have

$$ac + d = 0 \text{ and } bci = 0. \tag{4}$$

Therefore, $bc = 0$. Since, by hypothesis, $(a + bi)$ is a complex number, $b \neq 0$. Hence, $c = 0$. Then from $ac + d = 0$ of (4), we get $d = 0$, and thus the remainder $cx + d$ in (2) is zero, and $D(x)$ is a factor of $f(x)$.

Corollary. *Any polynomial $f(x)$ with real coefficients can be expressed as a product of real linear and quadratic factors.*

Since imaginary factors of $f(x)$ occur in conjugate pairs when the coefficients in $f(x)$ are real, it follows that in this case $f(x)$ may be regarded as the product of a_0, of real linear factors of the type $(x - r)$, and quadratic factors of the type

$$(x - a)^2 + b^2 = (x - a - bi)(x - a + bi),$$

where a, b, and r are real numbers. When all the roots of $f(x) = 0$ are real, the polynomial $f(x)$ is the product of real linear factors, but if $f(x) = 0$ has imaginary or complex roots, $f(x)$ contains real quadratic factors of the type $(x - a)^2 + b^2$ which cannot be separated into real linear factors.

116. Graphs of $f(x)$ when some linear factors are imaginary. In Art. 114 the graph of $f(x)$ is discussed when the polynomial is the product of real linear factors, and it is shown that, corresponding to each linear factor $(x - r)$, the graph meets the X-axis at $x = r$. It should now be noted that

$$(x - a)^2 + b^2 > 0,$$

for all real values of x, and there is, therefore, corresponding to such quadratic factors of $f(x)$, no intersection of the graph with the X-axis.

Example: Graph
$$\begin{aligned} f(x) &= x^4 - 7x^3 - 4x^2 + 78x \\ &= x(x + 3)(x^2 - 10x + 26) \\ &= x(x + 3)[(x - 5)^2 + 1]. \end{aligned}$$

Corresponding to the linear factors x and $x + 3$, the graph intersects the X-axis at $x = 0$ and $x = -3$ respectively (Fig. 40). Corresponding to the quadratic factor $x^2 - 10x + 26$ there is no intersection with the X-axis. (In Fig. 40 one horizontal space represents one unit, while one vertical space represents twenty units.)

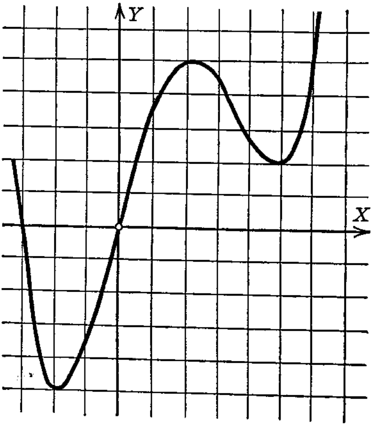

FIG. 40

EXERCISES

Plot the graph of each of the following:

1. $f(x) = (x - 1)(x - 2)(x - 3)$.
2. $f(x) = (x + 3)^2(x - 2)^2$.
3. $f(x) = (x - 1)^3(x + 2)$.
4. $f(x) = (x - 1)(x + 2)(x + 5)$.
5. $f(x) = (x - 1)(x - 2)^2$.
6. $f(x) = x(x - 3)^3$.

Separate the following polynomials into real linear and quadratic factors and plot the graphs.

7. $x^3 - 1$. **8.** $x^3 + 1$. **9.** $x^4 - 1$. **10.** $x^6 - 1$.

11. $x^4 + 4x^3 + 3x^2 - 4x - 4$.

12. Show that an equation $f(x) = 0$ of odd degree and with real coefficients has an odd number of real roots.

117. Transformation of an equation multiplying each root by a constant. The solution of an equation $f(x) = 0$ is often facilitated by transforming it into an equation whose roots are equal to those of the original equation times a constant. If we make $x = \frac{y}{m}$ (or $y = mx$) in $f(x) = 0$, we obtain an equation in y whose roots are m times those of $f(x) = 0$. In particular, if $m = -1$, we make $x = -y$ in $f(x) = 0$, *and obtain* $f(-y) = 0$ *whose roots are equal in absolute value but opposite in sign to those of* $f(x) = 0$.

These transformations can be performed rapidly by means of the following rules:

1. *To obtain an equation each of whose roots is m times a corresponding root of $f(x) = 0$: multiply the successive coefficients beginning with that of x^{n-1} by m, m^2, m^3, $\cdots$, respectively and replace x by y.**

* In carrying out this rule any missing power of x should be supplied with zero as a coefficient.

For example, to find the equation each of whose roots is double the roots of the equation $x^4 - 4x^3 + 3x^2 + 1 = 0$, we make $m = 2$, and obtain

$$y^4 - 2(4y^3) + 2^2(3y^2) + 2^3(0 \cdot y) + 2^4 = 0,$$
$$y^4 - 8y^3 + 12y^2 + 16 = 0.$$

To establish this rule, substitute $x = \dfrac{y}{m}$ in

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \quad (1)$$

The result of this substitution is

$$a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0, \quad (2)$$

or $a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \cdots + m^{n-1}a_{n-1}y + m^na_n = 0, \quad (3)$

after multiplying by the constant m^n. The rule is thus established.

2. *To obtain an equation each of whose roots is equal in absolute value to a root of $f(x) = 0$, but opposite in sign: change the signs of the odd degree terms in $f(x) = 0$ and replace x by y.*

For example, the roots of the equation

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$$

are $2, 4, -1, -3$, and the equation with roots $-2, -4, 1, 3$ is

$$y^4 + 2y^3 - 13y^2 - 14y + 24 = 0.$$

The rule follows at once from rule 1, by making $m = -1$.

EXERCISES

Obtain equations in y whose roots are equal to the roots of the following equations multiplied by the number opposite.

1. $x^3 + 2x^2 - 7x - 1 = 0.$ (5)
2. $x^3 - \dfrac{x^2}{5} - \dfrac{8}{25} = 0.$ (5)
3. $x^4 - 10x^2 - 3x - 2 = 0.$ (-1)
4. $x^3 - 3x^2 + 10 = 0.$ (-2)

Obtain equations in y whose roots are equal to the roots of the following equations multiplied by the smallest number which will make all the coefficients integers and that of the highest power unity.

5. $x^3 - 2x^2 + \dfrac{1}{3}x - 10 = 0.$
6. $x^3 + x^2 + \dfrac{x}{6} - \dfrac{83}{36} = 0.$
7. $3x^3 + 4 = 0.$
8. $x^3 - \dfrac{x^2}{3} - \dfrac{8}{9} = 0.$
9. $2x^3 - 3x^2 + 5x + 3 = 0.$
10. $5x^3 + 6 = 0.$

11. $108x^3 - 54x^2 + 45x - 13 = 0$. (See exercise 2, Art. 121.)

12. $12x^3 - 4x^2 - 3x + 6 = 0$.

13. Obtain equations in y whose roots are equal in absolute value but opposite in sign to the roots of equations given in exercises 1–4.

118. Descartes's rule of signs. In a polynomial arranged in descending powers of x, if two successive terms differ in sign there is said to be a **variation** in sign. For example,

$$x^4 - 4x^3 + 3x^2 + 4x - 5$$

has three variations of sign as is shown more clearly by writing down the signs $+ - + + -$. Multiply this polynomial by $(x - 1)$. There results

$$x^5 - 5x^4 + 7x^3 + x^2 - 9x + 5$$

with four variations of sign. This last polynomial has one more positive zero (see Art. 114) than the first. If increasing the number of positive zeros of a polynomial always increases the number of variations in sign by at least one, then the number of positive roots is never greater than the number of variations of sign.

THEOREM. *The number of positive roots of an equation $f(x) = 0$ does not exceed the number of variations of sign of $f(x)$, nor does the number of negative roots exceed the number of variations of sign of $f(-x)$.*

This is Descartes's rule of signs.

The part which relates to positive roots will be established by showing that whenever a positive root r is introduced into an equation, there is added at least one variation of sign. Let $f(x) = 0$ be an equation of degree m. It is only necessary to show that $(x - r)f(x)$ has at least one more variation of sign then $f(x)$. Group the terms of $f(x)$ between consecutive variations of sign in brackets. In general, for a function of degree m, we have

$$\begin{aligned} f(x) = {} & [b_0x^m + b_1x^{m-1} + \cdots + b_px^{m-p}] \\ & - [b_{p+1}x^{m-p-1} + b_{p+2}x^{m-p-2} + \cdots + b_qx^{m-q}] \\ & + [b_{q+1}x^{m-q-1} + \cdots] \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \pm [b_{m-v}x^v + b_{m-v+1}x^{v-1} + \cdots + b_m], \end{aligned}$$

where $b_0, b_1, b_2, \cdots, b_m$ are positive.

Multiplying this function first by x, then by $-r$, and adding, we obtain

$$
\begin{aligned}
(x-r)f(x) = {} & [b_0x^{m+1} + (b_1 - b_0r)x^m \pm \cdots] \\
& - [(b_{p+1} + rb_p)x^{m-p} \pm \cdots] \\
& + [(b_{q+1} + rb_q)x^{m-q} \pm \cdots] \\
& \cdots\cdots\cdots\cdots \\
& \pm [(b_{m-v} + rb_{m-v-1})x^{v+1} \pm \cdots] \\
& \mp rb_m.
\end{aligned}
$$

It will be noticed that the coefficients of the first terms in the several brackets, that is, b_0, $(b_{p+1} + rb_p)$, $(b_{q+1} + rb_q)$, $\cdots$, are all positive. Hence, the signs between the brackets remain unchanged. The signs within the brackets are uncertain, but however they may occur there is at least one variation between the first term of one bracket and the first term of the next bracket. Hence, as far as the terms in the brackets are concerned the number of variations remains the same or is increased. But there is added the variation caused by the term $\mp rb_m$ whose sign differs from the sign of the last bracket. Therefore, there is at least one more variation in $(x - r)f(x)$ than in $f(x)$.

The part of Descartes's rule which relates to negative roots follows from the fact that the roots of $f(-x) = 0$ are equal in absolute value but opposite in sign to the roots of $f(x) = 0$.

EXERCISES

Without solving, find the maximum number of positive and of negative roots, and any other information about the nature of the roots by the use of Descartes's rule of signs and other theorems.

1. $x^3 + 5x - 7 = 0$.

Solution: There is one variation in sign, hence, there is not more than one positive root. $f(-x) = -x^3 - 5x - 7$ with no variation in sign, hence, there are no negative roots. Since there are three roots of the equation, two are imaginary and one positive.

2. $3x^4 + x^2 + 2 = 0$. *Hint:* Only even powers of x occur.

3. $x^3 + 2x + 4 = 0$.

4. $5x^4 + x^2 + 2 = 0$.

5. $3x^5 - 2x^2 - 5 = 0$.

6. $2x^4 + 3x^2 + 1 = 0$.

7. $3x^3 - 8x^2 - 5 = 0$.

8. $x^5 + 1 = 0$.

9. $x^3 + 3x^2 + 1 = 0$.

10. $x^6 + 4x^2 + x = 0$.

11. $x^n - 1 = 0$ (n odd).

12. $x^n - 1 = 0$ (n even).

13. Given that the roots of $5x^3 - 3x^2 - 4x + 11 = 0$ are all real, determine the signs of the roots.

14. Show that the equation $7x^6 - 2x^2 - 2x + 4 = 0$ has at least two imaginary roots.

15. Show that the equation $x^8 + 5x^3 + 4x - 10 = 0$ has six and only six imaginary roots.

The roots of the following equations being all real, determine their signs.

16. $x^4 - 10x^2 + 5 = 0$.

17. $2x^3 - 3x^2 - 17x + 30 = 0$.

18. $x^4 - 8x^3 + 17x^2 + 2x = 24$.

19. $x^3 - 6x^2 + 11x - 6 = 0$.

20. $x^5 + 2x^4 - 4x^2 + x + 2 = 2x^3$.

21. $x^4 - 58x^2 + 441 = 0$.

119. Useful upper and lower bounds for roots. In the present chapter, we are concerned with the graph of $f(x)$ chiefly throughout an interval on x that contains the real roots of $f(x) = 0$. To save unnecessary labor in plotting, it is desirable to know upper and lower bounds of such an interval.

If no real root of $f(x) = 0$ is greater than b, nor less than b_1, the number b is said to be an **upper bound** and the number b_1 a **lower bound** for real roots of $f(x) = 0$.

A useful upper bound can often be found by means of the following

THEOREM. *If b is positive or zero, and if each sum in the synthetic division of $f(x)$ by $(x - b)$ is positive or zero, then no real root of $f(x) = 0$ is greater than b.*

The theorem is fairly obvious, since a greater number than b would make the sums still greater. For example, to show that 6 is greater than any real root of

$$f(x) = x^4 - 5x^3 + 3x^2 - 42x - 50 = 0, \tag{1}$$

we divide $f(x)$ by $(x - 6)$ by synthetic division,

$$\begin{array}{rrrrr|l} 1 & -5 & +3 & -42 & -50 & \underline{6} \\ & 6 & +6 & +54 & +72 & \\ \hline 1 & +1 & +9 & +12 & +22 & \end{array}$$

and observe that a number greater than 6 would increase each sum.

To find a lower bound of the negative roots of $f(x) = 0$, it is only necessary to find as above, by synthetic division, an upper bound of the positive roots of $f(-x) = 0$.

For example, to find a lower bound for the roots of (1), divide

$$f(-x) = x^4 + 5x^3 + 3x^2 + 42x - 50$$

by $(x - 1)$. We thus find 1 to be an upper bound for roots of $f(-x) = 0$. Hence, -1 is a lower bound for roots of (1).

An upper or a lower bound obtained by the theorem of page 158 is not necessarily very close to a root.

For example, consider an upper bound for roots of

$$x^3 - 9x^2 + 23x - 15 = 0. \tag{2}$$

By our theorem, 9 is an upper bound, but 5 is actually the largest root of (2) as shown in exercise 1, Art. 121.

While we may thus cite some examples in which an upper bound obtained by our theorem is not very close to the largest root, the bounds obtained are often close and very useful.

120. Location theorem. *If $f(a)$ and $f(b)$ have contrary signs, the equation $f(x) = 0$ has at least one real root between a and b.*

Thus the points P_1 and P_2 (Fig. 41) which correspond to $x = a$ and $x = b$ are on opposite sides of the X-axis, and any continuous curve connecting P_1 and P_2 crosses the X-axis at least once between a and b. Since, to every intersection of the graph with the X-axis there corresponds a real root of the equation (Art. 114), we assume this theorem.

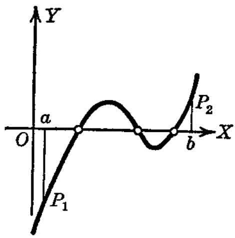

FIG. 41

EXERCISES

By the method of Art. 119, find integers that are upper and lower bounds of the roots of the equations.

1. $x^4 - 5x^3 + 3x^2 - 42x - 50 = 0.$
2. $x^4 - 2x^3 + 3x^2 - 5x + 1 = 0.$
3. $x^3 - 2x^2 - 35x + 14 = 0.$
4. $x^3 - 3x^2 - 2x + 5 = 0.$
5. $x^4 - 5x^3 - 8x + 1 = 0.$
6. $2x^3 - 3x^2 + 5 = 0.$
7. $2x^4 - 6x^3 + 4x + 6 = 0.$
8. $x^3 + 7x^2 - 29x + 13 = 0.$

9. By means of the location theorem show that the integer obtained in each of the exercises 1–4 is the smallest integer that is an upper bound for the roots.

Find the integral part of each real root of

10. $x^3 + x^2 - 2x - 1 = 0.$
11. $x^3 + 2x + 5 = 0.$
12. $x^3 - 2x + 5 = 0.$
13. $x^3 + 2x - 5 = 0.$
14. $x^4 - 12x^3 + 12x - 3 = 0.$
15. $8x^3 - 36x^2 + 46x - 15 = 0.$
16. $x^3 - 3x^2 - 4x + 11 = 0.$
17. $x^3 - 2x - 5 = 0.$

121. Theorem concerning rational roots. *If an equation*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (1)$$

with integral coefficients, has a rational root $\frac{b}{c}$, *where* $\frac{b}{c}$ *is in its lowest terms, then* b *is a factor of* a_n *and* c *is a factor of* a_0.

Since $\frac{b}{c}$ is in its lowest terms, it is implied that b and c are integers with no common factors except 1 and -1. Since $\frac{b}{c}$ is a root of (1), we have

$$a_0\frac{b^n}{c^n} + a_1\frac{b^{n-1}}{c^{n-1}} + a_2\frac{b^{n-2}}{c^{n-2}} + \cdots + a_{n-1}\frac{b}{c} + a_n \equiv 0. \quad (2)$$

Multiply (2) by c^n. This gives

$$a_0b^n + a_1b^{n-1}c + a_2b^{n-2}c^2 + \cdots + a_{n-1}bc^{n-1} + a_nc^n \equiv 0. \quad (3)$$

Subtract a_nc^n from each side of (3), and factor b from the remainder in the left side. This gives

$$b(a_0b^{n-1} + a_1b^{n-2}c + a_2b^{n-3}c^2 + \cdots + a_{n-1}c^{n-1}) \equiv -a_nc^n. \quad (4)$$

Since the left side of (4) is an integer with b as a factor, its right side, $-a_nc^n$, must have b as a factor. Since b has no common factor with c, it must be a factor of a_n.

Next, transpose all the terms of (3) except a_0b^n. This gives

$$a_0b^n = -c(a_1b^{n-1} + a_2b^{n-2}c + \cdots + a_{n-1}bc^{n-2} + a_nc^{n-1}). \quad (5)$$

Since the right side of (5) is an integer with a factor c, the left side must contain a factor c. Since c has no factor in common with b, it must be a factor of a_0.

Hence, if our original equation (1) with integral coefficients has any rational roots, they may be found by trials that consist in testing which, if any, of the set of fractions whose numerators are factors of a_n and whose denominators are factors of a_0, will satisfy the equation (1).

If the coefficient a_0 of the highest power of x in a rational integral equation is unity, the equation is often written in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$

and is said to be **expressed in the *p*-form.**

COROLLARY. *Any rational root of an equation in the p-form with integral coefficients is an integer and an exact divisor of* p_n.

This important corollary follows at once from the theorem, since $c = 1$ or -1 when it is a factor of $a_0 = 1$, and thus $\frac{b}{c}$ must be an integer.

To obtain the rational roots of an equation in the p-form with integral coefficients, it is only necessary to test whether the integers which are the exact divisors of p_n *satisfy the equation.**

EXERCISES

Find the rational roots by trial. If in the process of finding rational roots, the depressed equation is a quadratic, find all the roots whether they are rational or not.

1. $x^3 - 9x^2 + 23x - 15 = 0$.

Solution: By Descartes's rule of signs, this equation has no negative roots. Hence, we need try only 1, 3, 5, and 15. By synthetic division,

$$\begin{array}{l|l} 1 - 9 + 23 - 15 & \underline{1} \\ \underline{\quad + 1 - \;\; 8 + 15} & \\ 1 - 8 + 15 + \;\; 0 & \end{array}$$

The depressed equation is $x^2 - 8x + 15 = (x - 5)(x - 3) = 0$. Hence, 1, 3, and 5 are the roots.

2. $108x^3 - 54x^2 + 45x - 13 = 0$.

Solution: In the p-form this equation is

$$x^3 - \frac{1}{2}x^2 + \frac{5}{12}x - \frac{13}{108} = 0. \tag{1}$$

Transform (1) into an equation whose roots are six times those of (1). This gives

$$x^3 - 3x^2 + 15x - 26 = 0. \tag{2}$$

The rational roots of (2) divided by 6 give the rational roots of (1). By Descartes's rule, (2) has no negative roots. Hence, we need try only 1, 2, 13, 26. Depressing the equation,

$$\begin{array}{l|l} 1 - 3 + 15 - 26 & \underline{1} \\ \underline{\quad + 1 - \;\; 2 + 13} & \\ 1 - 2 + 13 - 13 & \end{array}$$

Hence, 1 is not a root.

$$\begin{array}{l|l} 1 - 3 + 15 - 26 & \underline{2} \\ \underline{\quad + 2 - \;\; 2 + 26} & \\ 1 - 1 + 13 + \;\; 0 & \end{array}$$

* If p_n is a number with many factors, this method is likely to become laborious. Similarly, if a_0 or a_n or both of them have a large number of factors, the method suggested directly by the theorem is likely to be laborious.

The depressed equation $x^2 - x + 13 = 0$ has roots $\frac{1}{2} + \frac{i\sqrt{51}}{2}$, and $\frac{1}{2} - \frac{i\sqrt{51}}{2}$.

Hence, 2 is the only rational root of (2) and $\frac{1}{3}$ is the only rational root of (1).

3. $2x^3 + 3x^2 - 2x - 3 = 0.$ (1)

Solution: By the theorem, Art. 121, if a rational number $\frac{b}{c}$ is a root, the values of b are limited to ± 1 and ± 3; and the values of c are limited to ± 1 and ± 2. The possible rational numbers we can form for trial roots are ± 1, $\pm \frac{1}{2}$, ± 3, $\pm \frac{3}{2}$.

By synthetic division of the left side of (1) by $(x - 1)$, we write

$$\begin{array}{rrrr|l} 2 & +3 & -2 & -3 & 1 \\ & 2 & 5 & 3 & \\ \hline 2 & 5 & 3 & 0 & \end{array}$$

Hence, 1 is a root, and the depressed $2x^2 + 5x + 3 = 0$ gives -1, and $-\frac{3}{2}$ for the other roots.

4. $2x^3 + 3x^2 - 4x + 1 = 0.$

5. $x^4 - 2x^3 - 3x^2 + 8x - 4 = 0.$

6. $x^3 - x^2 - 11x - 4 = 0.$

7. $x^4 - 2x^3 - 20x^2 - 21x - 18 = 0.$

8. $x^4 - 15x^2 + 10x + 24 = 0.$

9. $x^3 - 4x^2 + 2x - \frac{1}{8} = 0.$

10. $x^3 + 3x^2 = 4x + 12.$

11. $3x^3 + 8x^2 + x = 2.$

12. $x^3 - 3x^2 + \frac{11}{4}x - \frac{3}{4} = 0.$

13. $12x^3 - 4x^2 - 3x + 1 = 0.$

14. $24x^3 - 26x^2 + 9x - 1 = 0.$

15. $2x^3 + x^2 + 2x + 1 = 0.$

16. $x^4 + x^2 - 2x + 6 = 0.$

17. $4x^3 - 8x^2 + 5x - 1 = 0.$

18. $x^4 - 45x^2 + 40x + 84 = 0.$

19. $24x^4 - 4x^3 - 2x^2 - \frac{13x}{24} + \frac{1}{4} = 0.$

20. $4x^3 - 12x^2 - 27x + 19 = 0.$

21. $x^5 - 8x^4 + 15x^3 + 20x^2 + 48 = 76x.$

122. Approximations to an irrational root of $f(x) = 0$ by successive graphs.* The simple geometrical fact that a root of the equation $f(x) = 0$ is a value of x at which the graph of $y = f(x)$ meets the X-axis, enables us to use the location theorem of Art. 120 to find closer and closer approximations to an irrational root by selecting a and b (Art. 120) closer and closer together with the root to be found between them.

While this plan of approximation to a root is simple in principle, the procedure is likely to be found rather laborious. The student

* This section is designed especially for those who omit Horner's method. While this graphical method is by no means a full substitute for Horner's method, it is applicable to a broader class of equations as is shown in this article.

may well depend much on his own ingenuity in estimating a root from the graph. The plan can probably be made clearest by examples.

Example 1. Find a root of

$$x^3 + 3x - 20 = 0.$$

Solution: First examine the equation for rational roots (Art. 121). We find the equation has none. Next, form a table of values of the function $y = f(x) = x^3 + 3x - 20$,

$$x = -2, \ -1, \quad 0, \quad 1, \quad 2, \ 3, \cdots$$
$$y = -34, -24, -20, -16, -6, 16, \cdots$$

and plot the function. The first figure of the root is 2. Moreover, if we assume that the graph is approximately a straight line between the points $(2, -6)$ and $(3, 16)$ we estimate from the graph the approximate value 2.3 for the first two figures of the desired root. Testing this value we find

$$f(2.3) = -.933,$$
$$f(2.4) = 1.024.$$

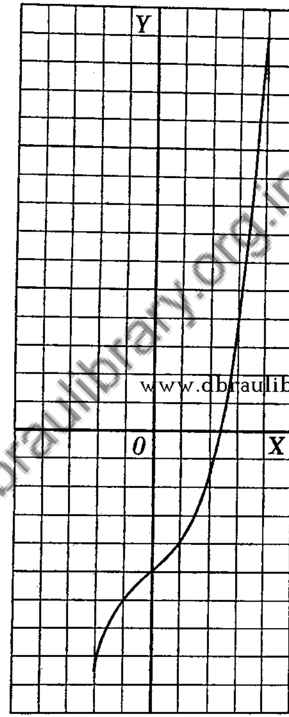

FIG. 42

Thus the root lies between 2.3 and 2.4 by Art. 120. By a repetition of this process, [plotting the points $(2.3, -.933)$, $(2.4, 1.024)$ on an *enlarged* scale tenfold that used originally] we infer that the root lies between 2.34 and 2.35. This judgment may be based on the fact that $f(2.3)$ is a little nearer 0 than $f(2.4)$, leading us to expect the root to be about halfway between 2.3 and 2.4 and nearer 2.3. By actual computation * we find

$$f(2.34) = -.167096,$$
$$f(2.35) = +.027875.$$

These figures suggest that the root is near 2.348. As

$$f(2.348) = -.011231808,$$

and

$$f(2.349) = +.008314559,$$

it is clear that the root to four figures is 2.348, but the exact value of the root is probably a little nearer to 2.349 than to 2.348. Thus, 2.349 – may be given as an approximation to the root.

It may be noted that the above method is likely to involve laborious arithmetical computations. On this account, some one

* These and later computations are greatly facilitated by having at hand a table of cubes.

of several other methods of approximating to the numerical roots of an equation may be preferable. One of these methods applicable to the rational integral equation is called Horner's method. This method is developed in Art. 124. However, any one of the methods of finding a close approximation to the numerical roots of equations is apt to be tedious. The method of this section has one advantage over Horner's method in that it is applicable to an equation $f(x) = 0$ when $f(x)$ is not a rational integral algebraic function as may be shown by the following:

Example 2. Find two real positive roots of

$$1 + (x - 2)^{\frac{1}{2}} - (x + 6)^{\frac{1}{3}} = 0 \text{ between 1 and 5.}$$

Form a table of values of $y = f(x) = 1 + (x - 2)^{\frac{1}{2}} - (x + 6)^{\frac{1}{3}}$.

$$\begin{aligned} x &= 2, \quad 3, \quad 4, \cdots. \\ y &= 0, \ -.08008, \quad .25978, \cdots. \end{aligned}$$

We note that $x = 2$ is a rational root, and that a root lies between 3 and 4. Since $f(3) = -.08008$ is less than one-third as much below 0 as $f(4) = .25978$ is above 0, we estimate 3.2 as the first two figures of the root although it is by no means sure that the first two figures are not 3.3.

To test this estimate, we find

$$f(3.2) = +.000066,$$
$$f(3.3) = +.0372317.$$

From the nearness of $f(3.2)$ to 0, we infer that $x = 3.2$ is a close approximation to the root, the exact root being slightly less than 3.2.

The method of this section may well be called the **method of successively enlarged scales.**

EXERCISES

1. Find three significant figures of the root of $x^3 - 9x + 3 = 0$ between 2 and 3 by the method of successively enlarged scales.

2. Find three significant figures of a root of $x^3 + 4x - 7 = 0$.

3. Find a root of $2x^3 + 3x^2 - 4x - 10 = 0$, correct to two decimal places.

4. The equation $1 + 2\sqrt{x - 1} - 2\sqrt[3]{x + 5} = 0$ has a root between 1 and 5. Find the first three significant figures of the root.

5. Find a root of $(7x^3 + 4x^2)^{\frac{1}{3}} + [10x(2x - 1)]^{\frac{1}{2}} - 28 = 0$, between 4 and 5 correct to two decimal places.

123. Transformation to diminish the roots. *To obtain an equation each of whose roots is less by h than a corresponding root of a given equation $f(x) = 0$: divide $f(x)$ by $(x - h)$ and indicate the remainder by R_n. Divide the quotient by $(x - h)$, and indicate the* *remainder by R_{n-1}. Continue this process to n divisions. The last quotient, a_0, and the remainders, R_1, R_2, $\cdots$, R_n are the coefficients of the transformed equation. The new equation is then,*

$$a_0y^n + R_1y^{n-1} + R_2y^{n-2} + \cdots R_{n-1}y + R_n = 0.$$

The division should be performed by the method of synthetic division.

For example, find the equation each of whose roots is less by 2 than the roots of the equation

$$x^3 - 4x^2 - 3x + 2 = 0.$$

The work is as follows:

$$\begin{array}{l|l}
1 - 4 - 3 + \ 2 & \underline{2} \\
\quad + 2 - 4 - 14 & \\
\hline
1 - 2 - 7 \,|\, - 12 & R_3 = -12, \\
\quad + 2 \quad\ 0 \,| & \\
\hline
1 + 0 \,|\, - 7 & R_2 = -7, \\
\quad\ \ 2 \,| & \\
\hline
1 + 2 & R_1 = 2, \\
 & a_0 = 1.
\end{array}$$

The required equation is

$$y^3 + 2y^2 - 7y - 12 = 0.$$

To establish the rule, substitute $x = y + h$ in

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{1}$$

This gives the equation in y

$$a_0(y + h)^n + a_1(y + h)^{n-1} + \cdots + a^{n-1}(y + h) + a_n = 0 \tag{2}$$

whose roots are less by h than those of (1). Expanding the binomial powers and arranging in powers of y, we may present the result in the form

$$a_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots + A_{n-1}y + A_n = 0. \tag{3}$$

If in (3), we make $y = x - h$, we obtain

$$a_0(x - h)^n + A_1(x - h)^{n-1} + A_2(x - h)^{n-2} + \cdots$$
$$\cdots + A_{n-1}(x - h) + A_n = 0. \tag{4}$$

which is the same as equation (1) arranged in powers of $x - h$. From the form of equation (4), it follows that A_n is the remainder when $f(x)$ is divided by $x - h$; A_{n-1} is the remainder when the quotient of the last-named division is divided by $x - h$; continuing this process to n divisions, A_1 is the last remainder, and a_0 is the last quotient. That is,

$$\begin{aligned} A_n &= R_n, \\ A_{n-1} &= R_{n-1}, \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ A_1 &= R_1, \end{aligned}$$

which establishes the rule.

EXERCISES

Obtain equations in y whose roots are equal to the roots of the following equations diminished by the number opposite.

1. $f(x) = 2x^4 - 3x^2 + 4x - 5 = 0.$ (2)

Solution: We apply synthetic division to divide $f(x)$ by $(x - 2)$ to get the coefficients of the equation in y as explained in Art. 123. Thus, we have

$$\begin{array}{rrrrrl|l} 2 & +\ 0 & -\ 3 & +\ 4 & -\ 5 & \underline{|2} & \\ & +\ 4 & +\ 8 & +\ 10 & +\ 28 & & \\ \hline 2 & +\ 4 & +\ 5 & +\ 14 & +\ 23 & & R_4 = 23, \\ & +\ 4 & +\ 16 & +\ 42 & & & \\ \hline 2 & +\ 8 & +\ 21 & +\ 56 & & & R_3 = 56, \\ & +\ 4 & +\ 24 & & & & \\ \hline 2 & +\ 12 & +\ 45 & & & & R_2 = 45, \\ & +\ 4 & & & & & \\ \hline 2 & +\ 16 & & & & & R_1 = 16, \\ & & & & & & a_0 = 2. \end{array}$$

Hence, $2y^4 + 16x^3 + 45x^2 + 56x + 23 = 0$ is the required equation.

2. $x^3 - 7x + 7 = 0,$ (1)
3. $x^3 - 27x - 36 = 0,$ (3)
4. $x^5 - 6x^4 + 7.4x^3 + 7.92x^2 - 17.872x - 0.79232 = 0,$ (1.2)
5. $2x^4 + 16x^3 + 45x^2 + 56x + 23 = 0,$ (− 2)
6. $x^3 + 3x^2 - 4x + 1 = 0,$ (1)
7. $x^3 + 20x^2 + 131x + 280 = 0,$ (− 5)
8. $x^3 + 6x^2 + 9x - 8 = 0,$ (− 2)
9. $x^3 + 5x^2 - 3 = 0,$ (0.7)
10. The roots of the equation $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$ are 2, 4, − 1, − 3. By the method of Art. 123, obtain an equation in y with roots 0, 2, − 3, − 5.

124. Irrational roots. Horner's method. The irrational roots of a numerical equation can be obtained to any desired number of decimal places by a method of approximation called **Horner's method.** The method is based mainly on successive diminutions of the roots of the equations to be solved (Art. 123). It can be best explained by first applying it to an example. In case an equation has some rational roots, it should always be depressed by removing such roots before considering irrational roots.

Example: Find the real roots of

$$x^4 - 2x^3 + 4x^2 - 15x + 14 = 0. \tag{1}$$

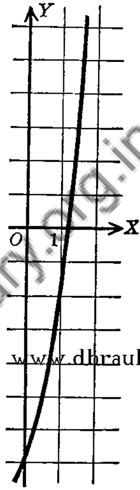

FIG. 43

1. Test for rational roots as in Art. 121. It results that 2 is the only rational root.

$$\begin{array}{rrrrr|l} 1 & -\,2 & +\,4 & -\,15 & +\,14 & \underline{2} \\ & +\,2 & +\,0 & +\,8 & -\,14 & \\ \hline 1 & +\,0 & +\,4 & -\,7 & +\,0 & \end{array}$$

The depressed equation is

$$x^3 + 4x - 7 = 0. \tag{2}$$

2. Test for the interval which contains the real roots. From Descartes's rule, equation (2) has not more than one positive root, and it has no negative root. Furthermore, 2 is greater than any root (Art. 119).

3. Plot $x^3 + 4x - 7$ from $x = 0$ to $x = 2$.

The graph (Fig. 43) shows that 1 is the first figure of the root.

4. Transform to diminish roots by 1; or graphically, change the origin to the point marked 1. The numerical work is as follows:

$$\begin{array}{rrrr|l} 1 & +\,0 & +\,4 & -\,7 & \underline{1} \\ & +\,1 & +\,1 & +\,5 & \\ \hline 1 & +\,1 & +\,5 & -\,2 & \\ & +\,1 & +\,2 & & \\ \hline 1 & +\,2 & +\,7 & & \\ & +\,1 & & & \\ \hline 1 & +\,3 & & & \end{array}$$

The first transformed equation is then

$$x_1^3 + 3x_1^2 + 7x_1 - 2 = 0. \tag{3}$$

This equation has a root between 0 and 1, since (2) has a root between 1 and 2. By evaluating $f(x_1) = x_1^3 + 3x_1^2 + 7x_1 - 2$ for successive tenths (0.0, 0.1, 0.2, $\cdots$, 0.9), we find that this function is negative when $x_1 = 0.2$ and positive when $x_1 = 0.3$. Hence, (3) has a root between 0.2 and 0.3. An approximation to this root is given by neglecting the second and third degree terms in (3) and solving

$$7x_1 - 2 = 0.$$

The root of this equation between 0 and 1 is $x_1 = 0.2 \cdots$. It is important to observe from the graph of $f(x)$ that the sign of the known term in each transformed equation is to be the same as that of the original equation after the rational roots have been removed.

Transforming (3) into an equation whose roots are less by 0.2, we have

$$\begin{array}{llll|l}
1 & 3 & 7 & -2 & \underline{0.2} \\
 & 0.2 & 0.64 & 1.528 & \\
\hline
1 & 3.2 & 7.64 & -0.472 & \\
 & 0.2 & 0.68 & & \\
\hline
1 & 3.4 & 8.32 & & \\
 & 0.2 & & & \\
\hline
1 & 3.6 & & &
\end{array}$$

or

$$x_2^3 + 3.6x_2^2 + 8.32x_2 - 0.472 = 0 \tag{4}$$

as the second transformed equation. The root of equation (4) which we seek lies between 0 and 0.1. Neglecting powers of x_2 higher than the first, it appears from the equation

$$8.32x_2 - 0.472 = 0$$

that x_2 lies between 0.05 and 0.06. That the root is in this interval may be tested by evaluating $x_2^3 + 3.6x_2^2 + 8.32x_2 - 0.472$ for $x_2 = 0.05$ and 0.06.

Transforming (4) by synthetic division into an equation whose roots are less by 0.05, we obtain

$$x_3^3 + 3.75x_3^2 + 8.6875x_3 - 0.046875 = 0. \tag{5}$$

Neglecting powers of x_3 higher than the first, it appears from the equation

$$8.6875x_3 - 0.046875 = 0$$

that x_3 lies between 0.005 and 0.006.

Transforming (5) by synthetic division into an equation whose roots are less by 0.005, we have

$$x_4^3 + 3.765x_4^2 + 8.725075x_4 - 0.003343625 = 0. \tag{6}$$

The root of this equation between 0 and 0.001 can be obtained at least as far as the first figure by neglecting powers of x_4 above the first. This gives

$$x_4 = 0.0003^+.$$

Transforming (6) into an equation whose roots are less by 0.0003, we obtain

$$x_5^3 + 3.7659x_5^2 + 8.72733427x_5 - 0.000725763623 = 0.$$

The root of this equation between 0 and 0.0001 can be obtained at least so far as the first significant figure by neglecting powers of x_5 above the first. This gives

$$x_5 = 0.00008^+.$$

Taking the sum of successive diminutions of the roots of (2), we obtain as the approximate value of the root sought

$$x = 1.25538^+.$$

The above computation is compactly arranged on page 170. The process can evidently be continued to find the root to any required number of decimal places.

If a root of an equation is known to be small, one important point to note is that such a root can, in general, be well estimated by dividing the known term, with its sign changed, by the coefficient of the first degree term. The coefficient of the first degree term is, for this reason, sometimes called the **trial divisor** in obtaining approximate roots. A still better estimate of a root can, in general, be obtained by dropping terms of degree higher than the second, and solving the quadratic.

When an equation has more than one irrational root, each is treated separately as we have treated the single irrational root in this example.

If two roots of an equation $f(x) = 0$ are nearly equal, their separation may become laborious, but the separation may be accomplished by assigning values to x sufficiently near each other in plotting the graph of $f(x)$. For example,

$$4x^3 - 24x^2 + 44x - 23 = 0,$$

has two roots between 2 and 3. By assigning successively the values $x = 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3$, in plotting the graph we find that one of these roots is between 2.2 and 2.3, while the other is between 2.8 and 2.9.

The following is a compact arrangement of the solution of the example of pages 167–9. The heavy type indicates the successive transformed equations.

1	**0**	**4**	**− 7**	**1.**
	1	1	5	
1	1	5	− 2	
	1	2		
1	2	7		
	1			
1	**3**	**7**	**− 2**	**0.2**
	0.2	0.64	1.528	
1	3.2	7.64	− 0.472	
	0.2	0.68		
1	3.4	8.32		
	0.2			
1	**3.6**	**8.32**	**− 0.472**	**0.05**
	0.05	0.1825	0.425125	
1	3.65	8.5025	− 0.046875	
	0.05	0.1850		
1	3.70	8.6875		
	0.05			
1	**3.75**	**8.6875**	**− 0.046875**	**0.005**
	0.005	0.018775	0.043531375	
1	3.755	8.706275	− 0.003343625	
	0.005	0.01880		
1	3.760	8.725075		
	0.005			
1	**3.765**	**8.725075**	**− 0.003343625**	**0.0003**
	0.0003	0.00112959	.002617861377	
1	3.7653	8.72620459	− 0.000725763623	
	0.0003	0.00112968		
1	3.7656	8.72733427		
	0.0003			
1	**3.7659**	**8.72733427**	**− 0.000725763623**	**0.00008**
	0.00008	0.0003012784	0.000698210843872	
1	3.76598	8.7276355484	− 0.000027552779128	

125. Negative roots. The negative roots of $f(x) = 0$ are obtained by finding the positive roots of $f(-x) = 0$, and changing their signs. It is therefore sufficient to discuss the method of obtaining positive roots.

126. Summary. In solving a numerical equation $f(x) = 0$ for all its real roots, the following procedure may be found helpful in systematizing the work:

1. *Test for rational roots; and if any exist, depress the equation by removing the corresponding factors.*

2. *Determine an interval which contains the positive irrational roots* (Art. 119).

3. *Plot the depressed polynomial to locate a root between consecutive integers. The smaller integer is the integral part of the root.*

4. *To approximate more closely to the root, apply either the method of successively enlarged graphs* (Art. 122), *or Horner's method* (Art. 124). *If Horner's method is chosen, the following summary of steps is likely to be helpful: fix the attention upon some positive root whose location is known to be between two consecutive integers. Obtain by synthetic division (Art. 123) an equation whose roots are less than those of the given equation by the smaller of these two integers. The new equation has a root between 0 and 1. Locate this root between two successive tenths; and decrease the roots by the smaller of these tenths. The equation thus obtained has a root between 0 and 0.1. Locate this root between two successive hundredths, and again decrease the roots by the smaller of these hundredths. Continue this process to any required number of decimal places.*

Add together all the diminutions of the roots to obtain the required root.

If more than one root is contained between two consecutive integers, separate them by means of the location principle.

5. *Treat negative roots in the same manner as positive roots after changing* $f(x) = 0$ *into* $f(-x) = 0$.

EXERCISES AND PROBLEMS

Find, by Horner's method, the prescribed root of each equation to two decimal places.

1. $x^3 - 3x^2 - 2x + 5 = 0$, root between 1 and 2.
2. $x^3 - 3x^2 - 2x + 5 = 0$, root between 3 and 4.
3. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$, root between 5 and 6.
4. $x^5 + 12x^4 + 59x^3 + 150x^2 + 201x - 207 = 0$, root between 0 and 1.

Find, to two decimal places, the prescribed roots using either the method of successively enlarged graphs, or Horner's method.

5. $3x^3 - 11x^2 + 6x + 7 = 0$, root between 2 and 3.
6. $x^3 + 10x^2 + 8x - 120 = 0$, root between 2 and 3.
7. $3x^3 + 14x^2 + 13x - 2 = 0$, root between 0 and 1.
8. $x^3 + 3x^2 - 4x + 1 = 0$, two roots between 0 and 1.

9. $x^4 + 10x - 100 = 0$, root between 2 and 3.

10. $x^3 - 9x - 5 = 0$, root between -3 and -2.

11. $x^4 + 10x - 100 = 0$, root between -4 and -3.

Find the rational roots, and the value of each irrational real root to two decimal places, by any method.

12. $x^3 - 100 = 0$.

13. $x^5 - 1000 = 0$.

14. $8x^3 - 12x^2 + 1 = 0$.

15. $x^3 + 4x^2 + 4x + 3 = 0$.

16. $x^4 - 3x^3 + 3 = 0$.

17. $x^3 - 3x - 1 = 0$.

18. $2x^4 - 12x^3 + 12x - 3 = 0$.

19. $3x^4 + 11 = 2x^3 + 21x^2 + 4x$.

20. $x^3 + 4x^2 = 5x + 20$.

21. $x^3 + 13 = 3x^2 + 4x$.

22. $x^3 + 3x^2 + 4x + 5 = 0$.

23. $x^3 + 30x = 420$.

24. $x^3 + 6 = 3x^2 + 2x$.

25. A sphere of yellow pine 1 foot in diameter floating in water sinks to a depth x given by

$$2x^3 - 3x^2 + 0.657 = 0.$$

Find the depth to three significant figures.

26. A sphere of ice 1 foot in diameter floating in water sinks to a depth x given by the equation

$$2x^3 - 3x^2 + 0.93 = 0.$$

Find the depth to three significant figures.

27. A cork sphere 1 foot in diameter floating in water sinks to a depth x given by the equation

$$2x^3 - 3x^2 + 0.24 = 0.$$

If the sphere is 2 feet in diameter, the immersed depth is given by

$$2x^3 - 6x^2 + 1.92 = 0.$$

Find the depths to two significant figures.

28. An open box is made of a rectangular piece of tin 10 inches by 20 inches by cutting equal squares from the corners and turning up the sides. Find (to two decimal places) the side of a square cut out if the volume of the box is 187 cubic inches.

29. A piece of property can be bought for \$7550 cash or \$8000, payable in four equal annual instalments of \$2000 each, the first instalment being paid at once, and the remaining instalments at the ends of 1, 2, and 3 years. What yearly rate of interest compounded annually gives the two offers equal present values?

30. The width of the strongest beam which can be cut from a log 12 inches in diameter is given by the positive irrational root of the equation

$$x^3 - 144x + 665 = 0.$$

Find the width to three significant figures.

31. The speed in feet per second of a 1-inch manila rope transmitting 4 horsepower, under a tension of 300 pounds on the tight side, is given by the equation

$$v^3 - 19200v + 211200 = 0.$$

Find the velocity to three significant figures.

32. The diameter of a water pipe whose length is 200 feet, and which is to discharge 100 cubic feet per second under a head of 10 feet, is given by the real root of the equation

$$x^5 - 38x - 101 = 0.$$

Find the diameter to three significant figures. (Merriman and Woodward, *Higher Mathematics*, p. 13.)

33. The algebraic treatment of the trisection of an angle whose sine is a involves the solution of the cubic equation

$$4x^3 = 3x - a.$$

The unknown, x, is the sine of one third the given angle. When $a = \frac{1}{2}\sqrt{2}$, find x to three significant figures.

34. A vat in the form of a rectangular parallelopiped is $8 \times 10 \times 12$ feet. If the volume is increased 500 cubic feet by equal elongations of the dimensions, find elongations in feet to two decimal places.

35. In problem 34, if the volume is increased by elongations proportional to the dimensions, find each elongation.

36. From the American Report on Wholesale Prices, Wages, and Transportation, for 1891, the median wage is given in dollars by $\frac{1}{4}$ of a value of x in the equation

$$2561\tfrac{1}{2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

where $a_0 = 6972\frac{91}{128}$, $a_1 = -657\frac{5}{48}$, $a_2 = -33\frac{43}{64}$, $a_3 = \frac{197}{48}$, $a_4 = -\frac{5}{128}$. Find the median wage correct to mills.

127. Coefficients in terms of roots. Let $r_1, r_2, \cdots, r_n$ be the roots of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0. \qquad (1)$$

Then, from Art. 112,

$$\begin{aligned} x^n &+ p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n \\ &= (x - r_1)(x - r_2) \cdots (x - r_n), \\ &= x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + (r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)x^{n-2} \\ &\quad - (r_1r_2r_3 + \cdots + r_{n-2}r_{n-1}r_n)x^{n-3} + \cdots + (-1)^n r_1r_2r_3 \cdots r_n, \qquad (2) \\ &\quad \cdots \cdots \cdots \end{aligned}$$

by actual multiplication of the binomial factors of the second member.

Equating coefficients in (2) (Art. 112, Cor. I), we have

$$\left.\begin{aligned} -p_1 &= r_1 + r_2 + \cdots + r_n, \\ p_2 &= r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n, \\ -p_3 &= r_1r_2r_3 + \cdots + r_{n-2}r_{n-1}r_n, \\ &\cdots\cdots\cdots \\ (-1)^n p_n &= r_1r_2r_3 \cdots r_n. \end{aligned}\right\} \qquad (A)$$

That is, $-p_1$ = sum of the roots,

p_2 = sum of products of roots taken two at a time,

$-p_3$ = sum of products of roots taken three at a time,

. . . .

$(-1)^n p_n$ = product of the roots.

If certain relations among the roots are given, the expressions (A) of the coefficients in terms of the roots may aid in solving the equation.

Example 1. The roots of $x^3 - 6x^2 + 11x - 6 = 0$ are in A.P. Find them.

Solution: Since the roots, r_1, r_2, r_3 are in A.P., we may set

$$r_1 = b - d,\quad r_2 = b,\quad r_3 = b + d, \tag{1}$$

where $b - d$ is the first term and d the common difference in the A.P.

By (A), Art. 127, from (1)

$$p_1 = -6 = -(r_1 + r_2 + r_3) = -3b, \tag{2}$$

and

$$p_3 = -6 = -r_1r_2r_3 = -(b - d)b(b + d). \tag{3}$$

From (2) and (3), we obtain $b = 2$, $d = \pm 1$, and the roots are 1, 2, 3.

It is sometimes desirable to transform a given equation into an equation in which the coefficient of the term of degree next to the highest is zero.

Example 2. Transform $4x^3 + 24x^2 - x - 27 = 0$ into an equation in which the second degree term is missing.

$$\begin{array}{rrrr|l}
4 & 24 & -1 & -27 & \underline{-2} \\
 & -8 & -32 & 66 & \\
\hline
4 & 16 & -33 & 39 & \\
 & -8 & -16 & & \\
\hline
4 & 8 & -49 & & \\
 & -8 & & & \\
\hline
4 & 0 & & &
\end{array}$$

Solution: By Art. 127, the sum of the roots is $-\dfrac{24}{4} = -6$. In the required equation, the sum of the roots must be zero. Hence, the sum of the three roots of the given equation must be increased by 6. This will be accomplished by increasing each of the three roots by 2. By the method of Art. 123, we obtain in y the equation

$$4y^3 - 49y + 39 = 0.$$

EXERCISES

By the use of equalities (A), Art. 127, write an equation with each of the following sets of prescribed roots.

1. $-1, 1, 2.$
2. $-2, -1, 4.$
3. $1, 3, 5.$
4. $-\frac{2}{3}, -1 + \sqrt{2}, -1 - \sqrt{2}.$

For each of the following incomplete equations find the root that is not given, and write the complete equation.

5. $x^3 - 4x^2 + \cdots = 0$, if 1 and 2 are roots.

6. $4x^3 - 6x^2 + \cdots = 0$, if 1 and $-\frac{1}{2}$ are roots.

7. $x^4 + \cdots + 4 = 0$, if 1, 2, and -1 are roots.

Transform each of the following equations into an equation with the next to the highest degree term missing.

8. $2x^3 - 6x^2 + 3x + 1 = 0$.

9. $2x^4 - 16x^3 + 25x^2 + 3x - 4 = 0$.

10. Given that $-1 + \sqrt{2}$ is one root of $3x^3 + 8x^2 + x - 2 = 0$, find the remaining roots.

11. Solve $x^3 - 2x^2 - 4x + 8 = 0$, the sum of two of the roots being 4.

12. Solve $2x^3 - 3x^2 + 2x = 3$, the sum of two of the roots being zero.

13. The roots of $x^3 - 6x^2 + 3x + 10 = 0$ are in A.P. Find them.

14. Solve $x^3 - 8x^2 + 5x + 50 = 0$, two of the roots being equal.

15. The roots of $x^3 - 7x^2 + cx - 8 = 0$ are in G.P. Find them, and the coefficient c.

128. Algebraic solution of equations. In Arts. 121–126, methods are discussed by which we obtain approximately the real roots of numerical equations. We turn now to a brief consideration of equations with literal coefficients.

Solving such an equation consists in obtaining an expression in terms of the coefficients which satisfies the equation. In other words, it consists in finding a formula which gives the roots in terms of the coefficients. For example, the roots of the typical quadratic

$$ax^2 + bx + c = 0,$$

are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots of an equation are functions of the coefficients, and it is important to inquire into the character of these functions. The solution is said to be an **algebraic solution,** if these functions of the coefficients involve no operations, except a finite number of additions, subtractions, multiplications, divisions, and extractions of roots.

The algebraic solution of an equation is often called the **solution by radicals.**

In Arts. 129, 130, the general cubic

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

and the general quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

are solved by radicals.

The algebraic solution of the general fifth degree equation

$$a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$$

engaged the attention of mathematicians during the eighteenth and the first quarter of the nineteenth century. In 1826 Abel proved that the typical fifth degree equation has no algebraic solution. Since that time a branch of mathematics, known as the theory of substitution groups, has been much developed. While a treatment of substitution groups is beyond the scope of this book, it may be stated that, by means of this theory, it is shown that no typical equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

has an algebraic solution if n exceeds 4; and necessary and sufficient conditions that an equation has an algebraic solution are established.

129. The cubic equation. The general cubic equation is

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0. \tag{1}$$

By making

$$x = y - \frac{a_1}{3a_0}, \tag{2}$$

equation (1) is transformed into

$$a_0y^3 + \left(a_2 - \frac{a_1^2}{3a_0}\right)y + \frac{2a_1^3}{27a_0^2} - \frac{a_1a_2}{3a_0} + a_3 = 0,$$

or

$$y^3 + \frac{3a_0a_2 - a_1^2}{3a_0^2}y + \frac{2a_1^3}{27a_0^3} - \frac{a_1a_2}{3a_0^2} + \frac{a_3}{a_0} = 0, \tag{3}$$

which has no term of the second degree.

Let

$$3H = \frac{3a_0a_2 - a_1^2}{3a_0^2}, \tag{4}$$

and

$$G = \frac{2a_1^3}{27a_0^3} - \frac{a_1a_2}{3a_0^2} + \frac{a_3}{a_0}. \tag{5}$$

Then (3) takes the form,

$$y^3 + 3Hy + G = 0. \tag{6}$$

Now assume

$$y = u^{\frac{1}{3}} + v^{\frac{1}{3}}, \tag{7}$$

and

$$-H^3 = uv. \tag{8}$$

From (6), (7), and (8),

$$-G = u + v. \tag{9}$$

Eliminating v from (8) and (9), we have

$$u^2 + Gu - H^3 = 0, \tag{10}$$

and solving this quadratic in u, we find for a solution,

$$u = \frac{-G + \sqrt{G^2 + 4H^3}}{2}. \tag{11}$$

From (8) and (11) we have

$$v = -\frac{H^3}{u} = \frac{-G - \sqrt{G^2 + 4H^3}}{2}. \tag{12}$$

The double sign before the radical in the solution of the quadratic in u is omitted because taking the negative sign before the radical would simply interchange the values of u and v. Since

$$y = u^{\frac{1}{3}} + v^{\frac{1}{3}},$$

the three values of y are:

$$\left.\begin{aligned} y_1 &= u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}}, \\ y_2 &= wu^{\frac{1}{3}} - \frac{H}{wu^{\frac{1}{3}}}, \\ y_3 &= w^2u^{\frac{1}{3}} - \frac{H}{w^2u^{\frac{1}{3}}}, \end{aligned}\right\} \tag{13}$$

where $u^{\frac{1}{3}}$ is any one of the three cube roots of u, and w is a complex cube root of unity (Art. 102).

Exercise. Test the solution by substitution of these values of y in (6).

By means of (2) and (13), the roots of equation (1) are

$$\left.\begin{aligned} x_1 &= u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}} - \frac{a_1}{3a_0}, \\ x_2 &= wu^{\frac{1}{3}} - \frac{H}{wu^{\frac{1}{3}}} - \frac{a_1}{3a_0}, \\ x_3 &= w^2u^{\frac{1}{3}} - \frac{H}{w^2u^{\frac{1}{3}}} - \frac{a_1}{3a_0}. \end{aligned}\right\} \tag{14}$$

When the coefficients of the equation are real numbers, the numerical character of the roots depends upon the number under the radical sign in (11) and (12).

When $G^2 + 4H^3$ is negative, u is a complex number. In this case, to obtain y from (7) would involve the extraction of the cube root of complex numbers. As we have no general algebraic rule for extracting such a cube root, the case in which $G^2 + 4H^3$ is negative is called the **irreducible** case. These roots may, however, be obtained by a method involving trigonometry (see Art. 102). Even when $G^2 + 4H^3$ is positive, the solution presented above is not, in general, so well adapted to obtaining real roots of numerical equations as the methods of Arts. 121–126.

130. The quartic equation. The general quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

may be written in the p-form (Art. 121) as

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0. \tag{1}$$

Adding $(mx + b)^2$ to both members of (1), we have

$$x^4 + p_1x^3 + (p_2 + m^2)x^2 + (p_3 + 2mb)x + p_4 + b^2 = (mx + b)^2. \tag{2}$$

Assume the identity,

$$x^4+p_1x^3+(p_2+m^2)x^2+(p_3+2mb)x+p_4+b^2 \equiv \left(x^2+\frac{p_1}{2}x+q\right)^2. \tag{3}$$

Equating coefficients of like powers of x, we have

$$p_2 + m^2 = \frac{p_1^2}{4} + 2q, \tag{4}$$

$$p_3 + 2mb = p_1q, \tag{5}$$

$$p_4 + b^2 = q^2. \tag{6}$$

Eliminating m and b from (4), (5), and (6), we obtain

$$(p_1^2 + 8q - 4p_2)(q^2 - p_4) = (p_1q - p_3)^2, \tag{7}$$

or $8q^3 - 4p_2q^2 + (2p_1p_3 - 8p_4)q + 4p_2p_4 - p_1^2p_4 - p_3^2 = 0.$ (8)

This is a cubic in q. Since the general cubic is solved by radicals in Art. 129, we may assume a value of q known. When q is known, the values of m and b are obtained from (4) and (6). From (2) and (3), we have

$$\left(x^2 + \frac{p_1}{2}x + q\right)^2 = (mx + b)^2, \tag{9}$$

which is equivalent to the two quadratic equations

$$x^2 + \frac{p_1}{2}x + q - mx - b = 0,$$

and

$$x^2 + \frac{p_1}{2}x + q + mx + b = 0.$$

The solutions of these two quadratics give the four roots of (1).

EXERCISES

1. Solve $x^3 - 4x^2 + 6x - 4 = 0$ and verify the results by substitution.

Solution: Here $a_0 = 1$, $a_1 = -4$, $a_2 = 6$, $a_3 = -4$.

From (4) and (5), Art. 129,

$$G = -\frac{20}{27},\ H = \frac{2}{9}.$$

From (11), Art. 129

$$u = \frac{10 + 6\sqrt{3}}{27},$$

$$u^{\frac{1}{3}} = \frac{1 + \sqrt{3}}{3}.$$

From (14), Art. 129 the roots of the given equation are

$$2,\ 1 + i,\ 1 - i.$$

Substitution for x shows that each of these numbers satisfies the equation to be solved.

2. Solve $x^4 - 6x^3 + 12x^2 - 20x - 12 = 0.$ (1)

Solution: Adding $(mx + b)^2$ to both members of this equation gives

$$x^4 - 6x^3 + (12 + m^2)x^2 + (2mb - 20)x + b^2 - 12 = (mx + b)^2. \tag{2}$$

Assume the identity

$$x^4 - 6x^3 + (12 + m^2)x^2 + (2mb - 20)x + b^2 - 12 \equiv (x^2 - 3x + q)^2. \tag{3}$$

Equating coefficients, we obtain

$$12 + m^2 = 9 + 2q, \tag{4}$$

$$2mb - 20 = -6q, \tag{5}$$

$$b^2 - 12 = q^2. \tag{6}$$

Eliminating m and b from these three relations, we have the cubic

$$q^3 - 6q^2 + 42q - 68 = 0. \tag{7}$$

This cubic has a root $q = 2$. From (4), (5), and (6), the corresponding values of m^2, b^2, and mb are

$$m^2 = 1, \; b^2 = 16, \; mb = 4. \tag{8}$$

From (2), (3), and (8), $\quad (x^2 - 3x + 2)^2 = (x + 4)^2. \quad$ (9)

This equation is equivalent to the two quadratic equations

$$x^2 - 3x + 2 - (x + 4) = 0, \tag{10}$$

and

$$x^2 - 3x + 2 + x + 4 = 0. \tag{11}$$

The roots of (10) are $2 \pm \sqrt{6}$, and those of (11) are $1 \pm i\sqrt{5}$. These four values satisfy the given quartic.

Solve the following equations by the methods of Arts. 129, 130.

3. $x^3 - 2x^2 + 3 = 0$.

4. $x^4 + x^3 - x^2 = 7x + 6$.

5. $x^3 + 4x^2 + 4x + 3 = 0$.

6. $2x^3 + 1 = 5x^2$.

7. $x^4 + 2x^3 + x^2 + 3 = 0$.

8. $x^3 + 3x = 6x^2 + 18$.

9. $x^3 = x^2 + 4x + 6$.

10. $x^4 - 3x^2 + 6x = 2$.

CHAPTER XV

LOGARITHMS

131. Generalization of exponents. In Art. 34, a^x is defined when x is a positive integer. Thus, $4^5 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$. Also a meaning is obtained (Arts. 35–39) from the laws of exponents for a^x when x is any rational number. Thus, $8^{\frac{2}{3}}$ is the square of the cube root of 8. But no meaning has been obtained for a^x when x is an irrational number. For example, $4^{\sqrt{2}}$ is thus far undefined. But approximations to $\sqrt{2}$ are given by the sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots.$$

If these successive decimal approximations to $\sqrt{2}$ are used as exponents, closer and closer approximations to $4^{\sqrt{2}}$ are obtained. If we write the sequence

$$4^1, 4^{1.4}, 4^{1.41}, 4^{1.414}, 4^{1.4142} \ldots$$

we can have as close an approximation to $4^{\sqrt{2}}$ as we please.*

In this chapter we shall assume that a^x (a positive) has a meaning when x is irrational and that the laws of exponents may be used for all *real* values of the exponents, *rational* or *irrational*.

132. Definition of a logarithm. If $a^x = y (a > 0, a \neq 1)$, *then x is said to be the* **logarithm** *of y to the base a*, and this is written $x = log_a\, y$.

The two equations $\qquad a^x = y \qquad (1)$

and $\qquad x = log_a\, y \qquad (2)$

are two ways of expressing the same thing, i.e., the exponent applied to a to give y is equal to x. The number a is called the **base** of the system of logarithms.

We shall assume in what follows:

Corresponding to any two positive numbers y and a ($a \neq 1$) there exists one and only one real number x such that $a^x = y$.

* If x is an irrational number and a variable z takes on a sequence of rational values approaching x as a limit, it may be proved in more advanced mathematics that $a^z (a > 0)$ has a limit equal to a^x.

This assumption is sometimes expressed by saying that any positive number has one and only one logarithm, whatever positive number is the base (unity excepted).

EXERCISES

1. $\log_2 8 = ?$ $\log_3 27 = ?$ $\log_{10} 1 = ?$ $\log_a a = ?$ $\log_9 3 = ?$

2. Find x in the following:

$\log_x 8 = 3,$ $\log_8 2 = x,$ $\log_2 x = 5,$ $\log_x 1000 = 3,$ $\log_{10} x = 5.$

3. Fill out the following table:

Base	Number	Logarithm
	49	2
3	$\frac{1}{81}$	
81		$-\frac{1}{2}$
	.01	-2
43	1	

133. Derived properties of logarithms.

1. *The logarithm of a product equals the sum of the logarithms of its factors.*

Let $\log_a u = x$ and $\log_a v = y,$ (1)

then, $a^x = u,\ a^y = v,$ (Definition of logarithm.)

and $uv = a^{x+y}.$ (Art. 34 and Art. 131.)

Hence, $\log_a uv = x + y,$

that is, $\boldsymbol{\log_a uv = \log_a u + \log_a v.}$

Similarly, $\boldsymbol{\log_a (uvw) = \log_a u + \log_a v + \log_a w,}$

and so on for any number of factors.

Example: $\log_{10} 255 = \log_{10} 3 + \log_{10} 5 + \log_{10} 17.$

2. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

As above, let $\log_a u = x$ and $\log_a v = y,$

then, $a^x = u,\ a^y = v,$

and $$\frac{u}{v} = a^{x-y}.$$

Hence, $$\log_a \frac{u}{v} = x - y,$$

that is, $$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

Example: $$\log_{10} \frac{625}{133} = \log_{10} 625 - \log_{10} 133.$$

3. *The logarithm of u^v is equal to v multiplied by the logarithm of u.*

To prove this, let $x = \log_a u$ or $a^x = u$. (1)

Then, from (1), $u^v = a^{vx}$.

Hence, $$\log_a u^v = vx = v \log_a u. \quad (2)$$

Example: $$\log_{10} (257)^{\frac{1}{2}} = \frac{1}{2} \log_{10} 257.$$

Making $v = n$ and $v = \frac{1}{n}$ respectively, we have

(*a*) *The logarithm of the nth power of a number is the logarithm of the number multiplied by n.*

(*b*) *The logarithm of the real positive nth root of a number is the logarithm of the number divided by n.*

EXERCISES

Express the logarithms of the following expressions in terms of the logarithms of integers.

1. * $\log \dfrac{\sqrt[4]{8}}{9^{\frac{1}{5}}6^{\frac{2}{3}}}$.

Solution: $$\log \frac{\sqrt[4]{8}}{9^{\frac{1}{5}}6^{\frac{2}{3}}} = \log \sqrt[4]{8} - \log 9^{\frac{1}{5}} - \log 6^{\frac{2}{3}} \qquad \text{(1 and 2, Art. 133.)}$$
$$= \frac{1}{4} \log 8 - \frac{1}{5} \log 9 - \frac{2}{3} \log 6. \qquad \text{(3, Art. 133.)}$$

2. $\log \dfrac{2^3}{7^2}$. **3.** $\log \dfrac{\sqrt{7}}{2^2\sqrt[3]{5}}$. **4.** $\log \dfrac{6^4 \cdot 5^2}{3^{\frac{1}{3}}}$.

Express the logarithms of the following in terms of the logarithms of prime numbers.

5. $\log \dfrac{(25)^{\frac{1}{3}}}{(30)^{\frac{1}{2}}}$. **6.** $\log \dfrac{(12)^2 \cdot \sqrt{6}}{(14)^3}$. **7.** $\log \sqrt{2}\,\sqrt[3]{3}\,\sqrt[4]{4}$. **8.** $\log 7!$ **9.** $\log \dfrac{8!}{3!5!}$

10. Prove $\log_a 1 = 0$ and $\log_a a = 1$.

Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, (see table, pages 190, 191), find the logarithms of the following numbers to the base 10.

* When in a problem the same base is used throughout, it is customary not to write the base.

11. 21.
12. 49.
13. 63.
14. $\sqrt{63}$.
15. 5.
16. 210.
17. $\sqrt{\frac{1}{2}}$.
18. $\sqrt{343}$.
19. $\dfrac{25}{49}$.
20. $\dfrac{128}{105}$.
21. $\sqrt[3]{252}$
22. $\sqrt[4]{400}$.
23. $\sqrt[5]{5000}$.
24. $\sqrt[3]{\frac{1}{3}}$.
25. $(2940)^{\frac{2}{3}}$.
26. $\sqrt{0.294}$.

134. Common logarithms. While any positive number can be used as the base of some system of logarithms, there are two systems in general use. These are the **common** or Briggs's system and the **natural** or Napierian system. In the common system the base is 10, while in the natural system the base is a certain irrational number $e = 2.71828\cdots$. It may be stated that the common system is adapted to numerical computation, while the natural system is adapted to analytical work.*

In the following discussion of common logarithms, $\log x$ is written as an abbreviation of $\log_{10} x$.

Since,

$$\begin{aligned} 10^0 &= 1 & 10^{-1} &= 0.1 \\ 10^1 &= 10 & 10^{-2} &= 0.01 \\ 10^2 &= 100 & 10^{-3} &= 0.001 \\ 10^3 &= 1000 & 10^{-4} &= 0.0001 \\ &\cdots & &\cdots \end{aligned}$$

it follows that

$$\begin{aligned} \log 1 &= 0 & \log 0.1 &= -1 \\ \log 10 &= 1 & \log 0.01 &= -2 \\ \log 100 &= 2 & \log 0.001 &= -3 \\ \log 1000 &= 3 & \log 0.0001 &= -4 \\ &\cdots & &\cdots \end{aligned}$$

So far as these powers of 10 are concerned, it may be observed that the logarithm of the number becomes greater as the number increases. In accordance with this observation, we may assume, if $a < x < b$, that

$$\log a < \log x < \log b. \tag{1}$$

For example, $\log 100 < \log 765 < \log 1000,$

or $2 < \log 765 < 3.$

* The notation $\ln x$ for $\log_e x$ and $\log x$ for $\log_{10} x$ is frequently used when both kinds of logarithms appear in the same problem.

When the logarithm of a number is not an integer, it may be represented at least approximately by means of decimal fractions. Thus, $\log 765 = 2.8837$ correct to four decimal places.

The integral part of a logarithm is called the **characteristic** and the decimal part is called the **mantissa.** In log 765, the characteristic is 2 and the mantissa is 0.8837. For convenience in constructing tables, it is desirable to select the mantissa as positive even if the logarithm is a negative number. For example, $\log \frac{1}{2} = -0.3010$; but since $-0.3010 = 9.6990 - 10$, this may be written $\log \frac{1}{2} = 9.6990 - 10$ with a positive mantissa. The following illustration shows the method of writing the characteristic and mantissa:

$$\begin{aligned}
\log 7185 &= 3.8564 \\
\log 718.5 &= 2.8564 \\
\log 71.85 &= 1.8564 \\
\log 7.185 &= 0.8564 \\
\log 0.7185 &= 9.8564 - 10 \\
\log 0.07185 &= 8.8564 - 10
\end{aligned}$$

135. Characteristic. With our decimal system of notation, the characteristic in the case of the base 10 is very easy to determine by a simple rule. Herein lies the advantage of this base.

If y is a number which has n digits in the integral part, then

$$10^{n-1} \leqq y < 10^n, \qquad (1)$$

and by Art. 134, (1), $n - 1 \leqq \log y < n$.

Hence, $\qquad \log y = n - 1 + \text{(a positive fraction)}$

or $\qquad \log y = \text{characteristic} + \text{mantissa}.$

Hence, *to find the characteristic of the common logarithm of a number which has an integral part, subtract* 1 *from the number of digits in the integral part.*

The logarithm of 0.1 is -1; hence, from (1) Art. 134, the logarithm of any number between 0.1 and 1 is some number between -1 and 0; that is, minus one plus the mantissa. For example,

$$\log 0.7185 = -1 + 0.8564 = 9.8564 - 10.$$

For a number between .01 and .1 the logarithm is minus two plus the mantissa. For example,

$$\log 0.07185 = -2 + 0.8564 = 08.8564 - 10.$$

This leads easily to the general rule for negative characteristics:

The characteristic of the common logarithm of any positive number less than 1 *is negative and numerically one greater than the number of zeros immediately following the decimal point.*

The result so obtained could manifestly also be obtained by the following rule:

To find the characteristic of the common logarithm of a decimal fraction, subtract from 9 *the number of ciphers between the decimal point and the first significant figure. From the number so obtained subtract* 10.

If two numbers contain the same sequence of figures, and therefore differ only in the position of the decimal point, the one number is the product of the other and an integral power of 10, and hence, by Art. 133, the logarithms of the numbers differ only by an integer. Thus,

$$\begin{aligned} \log 3722 &= \log 37.22 + \log 100 \\ &= \log 37.22 + 2. \end{aligned}$$

Hence, *the mantissa of the common logarithm of a number is independent of the position of the decimal point.* In other words, the common logarithms of two numbers which contain the same sequence of figures differ only in their characteristics (Art. 134). Hence, tables of logarithms contain only the mantissas, and the computer must find the characteristics by the foregoing rules.

136. Approximate numbers. Most of the numbers used in this chapter are approximations. For example, when we read log 7.185 = 0.8564, it does not mean that log 7.185 is exactly 0.8564, but that 0.8564 gives the value of log 7.185 as nearly as can be done with four figures. The approximate number 0.8564 would be written in a four-place table for any number between the exact numbers 0.85635 and 0.85645, and is "correct to four significant figures," while log 7.185 = 0.85642677 is "correct to eight significant figures." (See footnote, page 68.)

137. Significant figures. In counting the number of significant figures, we usually consider the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, but under certain circumstances 0 may be significant. It is always so if it occurs between two other significant figures. For example, in the following logarithms, the zeros are significant, 2.5011, 1.9009. On the other hand, the zeros in 0.00379 are not significant but are used simply to locate the decimal point. Zeros at the end of a number may or may not be significant. If we say that the population of the United States is 130,000,000, the last zeros are not significant for we do not know just what numbers should be there, since we cannot count the population correct to a single person. Again, if measurements are taken to the nearest tenth of an inch and the length of a desk is put down as 60.0 inches, then both zeros are significant.

The position of the decimal point has no influence on the number of significant figures, for example, the numbers 576.35, 57.635, 0.057635, considered as approximate numbers are all correct to five significant figures.

138. Rejecting figures. It is often necessary to reject figures at the end of an approximate number. This rejection of figures is often called "rounding off" the number. For example, if we are working with four-place data, the last two figures in the number 0.376741 are unnecessary and we write simply 0.3767. However, if the first rejected figure is greater than 5 or 5 followed by figures not all zeros, the last unrejected figure should be increased by 1. Thus, it is clear that 0.7686 is a closer approximation to 0.768583 than is 0.7685. If the rejected figure is 5 or 5 followed by zeros, it is often customary among computers to increase the last unrejected number by 1 if it is an odd number, but to leave it unchanged if it is an even number. For example, 1.4865 becomes 1.486, but 0.839350 becomes 0.8394 when these numbers are cut down to four significant figures. If this rule is followed in a long piece of computation, the errors tend to compensate one another. This rule has been followed in working the problems in this book.

139. Computation with approximate numbers. The results of calculations based upon approximate numbers are ordinarily approximate numbers.

While it is beyond the scope of a college algebra to go far into the propagation of errors in computing with approximate numbers, we shall give, without proof, two rules that are rather generally adopted by computers and that are likely to put us on our guard against retaining useless figures in computing with approximate numbers.

Rule for addition. *In the sum of a given set of numbers of which at least one is an approximate number, it is seldom useful to retain more decimal places than are found in one of the approximate numbers with the least number of decimal places.*

Thus, in adding

$$\begin{array}{r} 3.1416 \\ 6.28 \\ 2.412 \\ 7.9 \\ \hline 19.7 \end{array}$$

in which 7.9 is an approximate number, we report the sum 19.7.

A similar rule holds with regard to subtraction.

Rule for multiplication and division. *In a product or quotient, it is seldom useful to retain more significant figures than are found in one of the given approximate numbers with the least number of significant figures.*

Thus, if 8.3 is an approximate number, we write $(8.3)(3.1416) = 26$, and report only two significant figures.

Although the above rules ordinarily give useful approximate results, it will be shown in some of the following exercises that the last figure is not the most accurate that could be given.

EXERCISES

1. Explain the difference between the approximate numbers 71.4, 71.40, 71.400 where the zeros are to be considered as significant.

2. Distinguish between the numbers 7, 7.0, 7.00, 7.000, assuming each digit a significant number.

3. To twelve significant figures $\pi = 3.14159265359$. Write π to eleven, to ten, to nine, $\cdots$ to two significant figures.

4. The value $\frac{22}{7}$ is often used for π. To what number of significant figures is this value equivalent?

5. Add the following numbers which are to be considered as the results of measurements: 31.5, 3.126, 25.4301, 0.438.

6. The product of the approximate numbers 3.17 and 7.98 may take on any value between what two numbers? What is the product according to the above rules?

7. The product of the three approximate numbers 0.37, 7.3, and 2.1 may take on any value between what two numbers? What is the product according to our rules?

8. Suppose in the fraction $\frac{75}{0.389}$ that the numerator is an exact number, and the denominator is an approximate number, find the extreme values which the fraction may represent. What is the quotient in decimal notation according to our rules?

9. If both numerator and denominator of the fraction $\frac{75.0}{0.389}$ are approximate numbers, find in decimal notation the extreme values which the fraction may represent. What is the quotient according to our rules?

10. There are 2.540005 centimeters in one inch. A board measures 14.7 feet in length measured to the nearest tenth of an inch. Express the length in meters. Between what two numbers does the result lie?

140. Use of tables. On pp. 190, 191, a "four-place" table of logarithms is given. In this table, the mantissas of the logarithms of all integers from 1 to 999 are recorded correct to four decimal places. "Five-place," "six-place," and "seven-place" tables are in common use, but this four-place table will serve for our present purposes.

Methods by which such a table can be made will be discussed after applying the logarithms found in the table to purposes of arithmetical calculation. In order to use the tables we must know how to take from the tables the logarithm of a given number, and how to take from the tables the number which has a given logarithm.

141. To find from the table the logarithm of a given number.

Example 1. Find the logarithm of 821.

Glance down the column headed N for the first two significant figures, then at the top of the table for the third figure. In the row with 82 and the column with 1 is found 9143.

Hence, log 821 = 2.9143.

Example 2. Find the logarithm of 68.42.

This number has more than three significant figures, so that its logarithm is not recorded in the table. It may, however, be obtained approximately from logarithms recorded in the table by a process of **interpolation.** In this process, it is assumed that to a small change in the number, there corresponds a change in the logarithm which is proportional to the change in the number. This assumption is called the **principle of proportional parts.** As in example 1, we find that the mantissas for 6840 and 6850 are 8351 and 8357, respectively. The difference between these two mantissas is 6. Since 6842 is two tenths of the interval from 6840 to 6850, by the principle of proportional parts, we add to 8351,

$$0.2 \times 6 = 1^+.$$

Hence, $$\log 68.42 = 1.8352.$$

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

142. To find from the table the number which corresponds to a given logarithm.

Example 1. Find the number whose logarithm is 2.4675. The mantissa 4675 is not recorded in the table, but it lies between the two adjacent mantissas 4669 and 4683 of the table. The mantissa 4669 corresponds to the number 293 and 4683 corresponds to 294. The number 4675 is $\frac{6}{14}$ of the interval from 4669 to 4683. By the principle of proportional parts, the number whose mantissa is 4675 is $2930 + \frac{6}{14} \times 10 = 2934^{+}$.

Hence, $$\log 293.4 = 2.4675.$$

Example 2. Find the number whose logarithm is 9.3025 − 10.

From the table,
$$\begin{aligned} \log 0.2000 &= 9.3010 - 10 \\ \log 0.2010 &= \underline{9.3032 - 10} \\ \text{Difference} &= 0.0022 \end{aligned}$$

$$(9.3025 - 10) - (9.3010 - 10) = 0.0015.$$

By the principle of proportional parts, the number is

$$0.2000 + \frac{15}{22} \times 0.0010 = 0.2007.$$

EXERCISES

Obtain, from the table, the common logarithms of the following:

1. 43.
2. 430.
3. 1.71.
4. 777.
5. 0.0846.
6. 66600.
7. 5483.
8. 1.247.
9. 32.41.
10. 0.03752.
11. 444.4.
12. 3.1416.

Obtain, by means of the table, the numbers whose common logarithms are the following:

13. 1.3118.
14. 2.3118.
15. 9.6191 − 10.
16. 8.9069 − 10.
17. 4.8293.
18. 3.5071.
19. 7.7727 − 10.
20. 1.6446.
21. 0.5946.

143. Computation by means of logarithms. The application of logarithms to shorten calculations depends upon the properties of logarithms given in Art. 133. By means of logarithms laborious multiplications and divisions may be replaced by additions and subtractions; and involution and evolution may be replaced by multiplication and division.

EXAMPLES

1. Find the value of $N = \dfrac{6.320 \times 8.674}{2.851}$ to four significant figures.

$$\begin{aligned} \log 6.320 &= 0.8007 \\ \log 8.674 &= \underline{0.9382} \\ \log (6.320)(8.674) &= 1.7389 \\ \log 2.851 &= \underline{0.4550} \\ \log N &= 1.2839 \\ N &= 19.23. \end{aligned}$$

In using logarithms, *much time is saved and the liability of error is decreased by making a so-called form for all the work before using the table at all.*

Thus, in Example 1, the "form" is *

$$\begin{aligned} \log 6.320 &= \\ \log 8.674 &= \\ \log (6.320)(8.674) &= \\ \log 2.851 &= \\ \log N &= \\ N &= \end{aligned}$$

2. Make a form for evaluating $N = \dfrac{(6.85)^{\frac{1}{2}}\sqrt[3]{8.542}}{\sqrt{65.27}}$.

$$\begin{aligned} \log 6.85 &= \\ \log 8.542 &= \\ \log 65.27 &= \\ \log (6.85)^{\frac{1}{2}} &= \\ \log (8.542)^{\frac{1}{3}} &= \\ \log [(6.85)^{\frac{1}{2}}(8.542)^{\frac{1}{3}}] &= \\ \underline{\log (65.27)^{\frac{1}{2}}} &= \\ \log N &= \\ N &= \end{aligned}$$

* The logarithm of the reciprocal of x is called the **cologarithm** of x and is written colog x. Since $\log 1 = 0$,

$$\text{colog } x = \log \frac{1}{x} = -\log x.$$

In a series of operations involving multiplications and divisions, we have both additions and subtractions if logarithms are used. These operations are all additions if cologarithms are introduced in the calculations. Example 1 could then be worked as follows:

$$\begin{aligned} \log 6.320 &= 0.8007 \\ \log 8.674 &= 0.9382 \\ \text{colog } 2.851 &= \underline{9.5450 - 10} \\ \log N &= 1.2839, \\ N &= 19.23 \end{aligned}$$

3. Evaluate $N = \sqrt[3]{-58.61}$ *

$$\begin{aligned} \log 58.61 &= 1.7680n \\ \log (58.61)^{\frac{1}{3}} &= 0.5893n \\ N &= -3.885. \end{aligned}$$

EXERCISES AND PROBLEMS

Compute to four significant figures by logarithms.

1. 88.76×977.7.
2. $(4.783)^3$.
3. $34.21 \times 76.73 \times 1.026$.
4. $\dfrac{8162}{99.31}$.
5. $\dfrac{2781 \times 3.413}{87.52 \times 8.243}$.
6. $\dfrac{2180 \times (-27.27)}{17.86 \times 0.0327}$.
7. $\dfrac{0.8371 \times 0.05631}{9.734 \times 19.85}$.
8. $(0.9192)^9$.
9. $\left(\dfrac{1}{2}\right)^{\frac{1}{2}}$.
10. $\sqrt[7]{-7.777}$.
11. $\sqrt[7]{-0.7777}$.
12. $\left(\dfrac{2}{3}\right)^{10}$.
13. $(0.37)^{0.37}$.

Solution:

$$\begin{aligned} \log 0.37 &= 9.5682 - 10 \\ \log (0.37)^{0.37} &= 0.37(9.5682 - 10) \\ &= 3.5402 - 3.7000) \\ &= -0.1598 \\ &= 9.8402 - 10 \\ (0.37)^{0.37} &= 0.6922. \end{aligned}$$

14. $(0.7)^{0.8}$.
15. $\sqrt{0.001} \times \sqrt[3]{0.0001} \times \sqrt[4]{0.00001}$.
16. $\dfrac{\sqrt[3]{3}}{\sqrt{2}}$.
17. $\sqrt[5]{9.999}$.
18. $(0.1)^{0.1}$.
19. $(10)^{2.7} \cdot \sqrt[3]{\dfrac{8.716}{14.37}}$.
20. $(100)^{0.01}$.
21. $(100)^{-0.01}$.
22. $\sqrt[5]{\dfrac{\pi}{(21)^3}}$.
23. $(-0.3333)^{-\frac{6}{7}}$.
24. $\dfrac{\log 3}{\log 2}$.
25. $\dfrac{\log 794}{\log 0.39}$.
26. $\dfrac{(\log 5)(\log 7)}{(\log 2)(\log 3)}$.
27. $\log \left(\dfrac{7.632}{0.947}\right) - \dfrac{\log 7.632}{\log 0.947}$.
28. $\sqrt{(7635)^2 - (4133)^2} = \sqrt{(7635 + 4133)(7635 - 4133)}$.
29. $\sqrt[3]{(9.637)^2 - (0.8463)^2}$.
30. $\sqrt[3]{\log 3.718} - \log \sqrt[3]{3.718}$.

* When a number is negative, find its logarithm without regard to sign, writing n after a logarithm that corresponds to a negative number so as to keep the negative sign in mind.

31. Estimate the weight of a cork ball 6 feet in diameter, then calculate its weight in pounds. A cork whose volume is 1 cubic inch weighs 0.139 ounces. $\left(\text{The volume of a sphere is } \frac{4}{3}\pi r^3.\right)$

32. First estimate and then calculate the increase in weight of the ball in problem 31 if the radius is increased 1 inch.

33. The time t of oscillation of a simple pendulum of length l feet is given in seconds by the formula

$$t = \pi\sqrt{\frac{l}{32.16}}.$$

Find the time of oscillation of a pendulum 3.826 feet long. (Take $\pi = 3.142$.)

34. What is the weight in tons of a solid cast-iron sphere whose radius is 2.728 feet, if the weight of a cubic foot of cast iron is 446.1 pounds?

35. Find the volume and surface of a sphere of radius 1.471 feet. (Surface of a sphere $= 4\pi r^2$.)

36. The stretch of a brass wire when a weight is hung at its free end is given by the relation

$$S = k\frac{mgl}{\pi r^2},$$

where m is the weight applied, $g = 980$, l is the length of the wire, r is its radius, and k is a constant. Find k for the following values: $m = 944.8$ grams, $l = 213.2$ centimeters, $r = 0.32$ centimeter, and $S = 0.060$ centimeter.

37. Find the length l of a wire which stretches 5.9 centimeters for a weight of 1825 grams hanging at its free end, the diameter of the wire being 0.064 centimeter, and $k = 98 \cdot 10^{-12}$.

38. The weight P in pounds which will crush a solid cylindrical cast-iron column is given by the formula

$$P = 98{,}920\frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches and l the length in feet. What weight will crush a cast-iron column 6 feet long and 4.3 inches in diameter?

39. For wrought-iron columns the crushing weight is given by

$$P = 299{,}600\frac{d^{3.55}}{l^2}.$$

What weight will crush a wrought-iron column of the same dimensions as that in problem 38?

40. The weight W of one cubic foot of saturated steam depends upon the pressure in the boiler according to the formula

$$W = \frac{P^{0.941}}{330.36},$$

where P is the pressure in pounds per square inch. What is W if the pressure is 380 pounds per square inch?

41. The diameter in inches of a connecting rod depends upon the diameter D of the engine cylinder, l the length of the connecting rod, and P the maximum steam pressure in pounds per square inch, according to Mark's formula

$$d = 0.02758\sqrt{D \cdot l \cdot \sqrt{P}}.$$

What is d when $D = 20$, $l = 72$, and $P = 200$?

42. The discharge of water from a triangular weir is given by

$$q = \frac{8c}{15}H^{\frac{5}{2}}\sqrt{2g},$$

where c is a constant 0.592, g is the acceleration due to gravity 32.2 feet per second, and H is the waterhead. Find q when $H = 0.3$ foot.

43. The number, n, of vibrations per second made by a stretched string is given by the relation

$$n = \frac{1}{2l}\sqrt{\frac{Mg}{m}},$$

where l is the length of the string, M the weight used to stretch the string, m the weight of one centimeter of the string, and $g = 980$. Find n, when $M = 6213.6$ grams, $l = 84.9$ centimeters, and $m = 0.00670$ gram.

44. What must be the weight per centimeter length of a wire which is 70.95 centimeters long and is stretched by a weight of 4406.5 grams, in order that it may vibrate 178 times per second?

45. The formula $y = ks^x g^{c^x}$, where $\log k = 5.03370116$, $\log s = -0.003296862$, $\log g = -0.00013205$, $\log c = 0.04579609$, gives the number living at age x in Hunter's *Makehamized American Experience Table of Mortality*. Find, to such a degree of accuracy as you can secure with a four-place table of logarithms, the number living (1) at age 10, (2) at age 30.

46. The Dutch are said to have paid \$24 to the Indians in 1626 for Manhattan Island. What would this \$24 amount to in 1940 if it had been placed at interest of 4 per cent (a) compounded annually; (b) compounded semiannually?

144. Change of base. *The logarithm of a number y to the base b is equal to the product of its logarithm to the base a and the logarithm of a to the base b.*

That is, $$\log_b y = \log_a y \cdot \log_b a. \tag{1}$$

Let $$u = \log_a y \text{ and } v = \log_b y. \tag{2}$$

Then, $$a^u = y,\ b^v = y, \tag{3}$$

and $$a^u = b^v. \tag{4}$$

$$a = b^{\frac{v}{u}}, \tag{5}$$

$$\frac{v}{u} = \log_b a,$$

$$v = u \log_b a. \tag{6}$$

From (2) and (6), $$\mathbf{\log_b y = \log_a y \log_b a.} \tag{7}$$

Example: $\log_{10} 128 = \log_2 128 \log_{10} 2.$

Since tables to the base 10 are usually available, by making $b = 10$ in (7) we may write

$$\mathbf{\log_a y = \frac{\log_{10} y}{\log_{10} a}} \tag{8}$$

which is useful in finding the logarithm of y to any base.

Example: $\log_7 127 = \dfrac{\log_{10} 127}{\log_{10} 7} = \dfrac{2.1038}{0.8451} = 2.4894.$

By making $y = b$ in (7), we obtain

$$1 = \log_a b \log_b a.$$

That is, $$\mathbf{\log_b a = \frac{1}{\log_a b}.} \tag{9}$$

The number $\log_b a$ is often called the **modulus** of the system of base b with respect to the system of base a.

In Art. 135, attention is called to the advantages of 10 for the base of a system of logarithms to be used in numerical calculations. For analytical purposes, as will appear in the calculus, it is convenient to use **natural** logarithms. This system has for its base an irrational number $e = 2.71828 \cdots$. In the chapter on Infinite Series, there will be given a series from which this approximation to e is obtained, and another series from which the logarithm of a number to the base e can be obtained to any number of decimal places. It turns out that

$$\log_e 10 = 2.3026,$$

and $$\log_{10} e = \frac{1}{\log_e 10} = 0.4343.$$

By (1), $$\log_{10} y = \log_e y \log_{10} e,$$
$$= 0.4343 \log_e y,$$

and $$\log_e y = 2.3026 \log_{10} y.$$

The number $\log_{10} e = 0.4343$ is the **modulus** (to four significant figures) of common logarithms with respect to natural logarithms.

EXERCISES

1. Given $\log_e 3 = 1.0986$, find $\log_{10} 3$ and compare the result with the value in the table on page 190.

2. Find $\log_e 2$, $\log_e 3$, $\log_e 11$, $\log_e 171$, $\log_e 0.5$.

Find the logarithms of the following numbers:

3. 10 to the base 5.

4. 10 to the base 3.

5. 3 to the base 2.

6. 2 to the base 3.

7. 3 to the base 6.

8. 700 to the base 7.

9. 800 to the base 8.

10. 10000 to the base 100.

11. 3 to the base $\frac{1}{2}$.

12. 10 to the base 0.3.

13. 0.536 to the base 2.

14. 0.0536 to the base 2.

145. Graph of $y = \log_a x (a > 1)$. A general notion of the value of the logarithm of any number can be easily fixed by reference to the graph of $y = \log_a x$. This graph is also the graph of $x = a^y$.

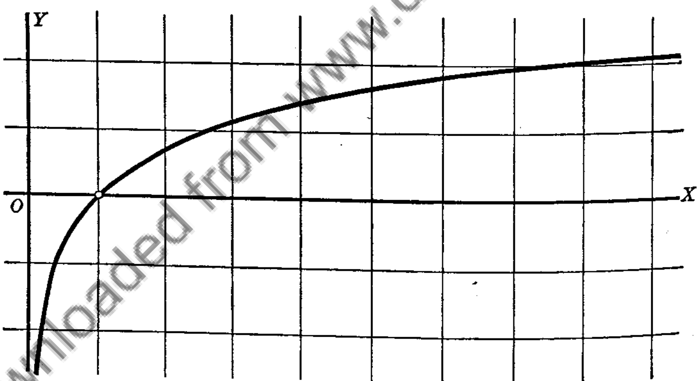

FIG. 44

In the graph (Fig. 44) we take $a = e = 2.718\cdots$, but the general form of the curve is not changed if a be given any other positive value greater than 1. If the student retains this picture, he should find it easy to keep in mind the following facts when the base is greater than unity.

1. A negative number does not have a real number for its logarithm.

2. The logarithm of a positive number is positive or negative according as the number is greater than or less than 1.

3. If x approaches zero, $\log x$ decreases without limit.
4. If x increases indefinitely, $\log x$ increases without limit.

EXERCISES

1. Plot the graph of $y = \log_{10} x$ by using tables to find $\log_{10} x$.
2. Plot the graph of $y = \log_5 x$. *Hint:* $\log_5 x = \dfrac{\log_{10} x}{\log_{10} 5}$.
3. Plot the graph of $x = \log_5 y$.
4. Plot the graph of $x = \log_2 y$.

146. Exponential and logarithmic equations. An equation which involves the unknown or unknowns in the exponents is often called an **exponential equation.** Thus, $2^x = 16$ is an exponential equation in x. In this simple example, the value of x can be obtained by inspection; but a table of logarithms is, in general, of value in solving exponential equations.

Such equations arise in a variety of problems. For example, the pressure of the atmosphere in pounds per square inch at a height of x feet is given approximately by the relation

$$P = P_0 e^{-kx}$$

where P_0 is the pressure at sea level and k is a constant.

Example: What is the pressure of the atmosphere per square inch at a height of one mile, given $k = 0.00003776$ and pressure at sea level, 14.72 pounds per square inch?

Solution: Let P be the pressure at 5280 feet, then

$$\begin{aligned} P &= 14.72e^{-kx}, \\ \log P &= \log 14.72 - kx \log e, \\ &= 1.1679 - 0.00003776 \cdot 5280 \cdot 0.4343, \\ &= 1.0813, \\ P &= 12.06 \text{ pounds per square inch.} \end{aligned}$$

Equations of this type also occur in certain compound interest problems. Examples will be found in Chap. XVI.

An equation which involves the logarithm of an expression that contains an unknown is sometimes called a **logarithmic** equation. Thus,

$$\log_{10} 2x = 3$$

is a logarithmic equation. To solve this equation, we may write, from the definition of a logarithm,

$$2x = 10^3 = 1000.$$

Hence,

$$x = 500.$$

EXERCISES AND PROBLEMS

Solve the following equations for x.

1. $5^x = 10$.

Solution: Since $5^x = 10$,

$$\log_{10} 5^x = \log_{10} 10 = 1.$$

$$x \log_{10} 5 = 1.$$

$$x = \frac{1}{\log_{10} 5}$$

$$= \frac{1}{.6990} = 1.431.$$

2. $2^{3x} 5^{2x-1} = 4^{5x} 3^{x+1}$.

Solution: $\log_{10} 2^{3x} 5^{2x-1} = \log_{10} 4^{5x} 3^{x+1}$,

$$3x \log_{10} 2 + (2x - 1) \log_{10} 5 = 5x \log_{10} 4 + (x + 1) \log_{10} 3$$
$$= 10x \log_{10} 2 + (x + 1) \log_{10} 3.$$

Transposing and collecting terms, we have

$$x(2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3) = \log_{10} 3 + \log_{10} 5.$$

$$x = \frac{\log_{10} 3 + \log_{10} 5}{2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3}$$

$$= \frac{0.4771 + 0.6990}{1.3980 - 2.1070 - 0.4771}$$

$$= -0.9916.$$

3. $16 = \log_{10} x^2$.

Solution: $16 = \log_{10} x^2$, (1)

From (1), $x^2 = 10^{16}$, (2)

$x = \pm 10^8$. (3)

4. $2^x = 5$.

5. $(0.2)^x = 0.5$.

6. $7^{2x-3} = 10$.

7. $8^{x^2} = 100$.

8. $2^{x^2-x} = 64$.

9. $3 \log_{10} x - 4 = 0$.

10. $3 \log_{10} x + 4 = 0$.

11. $(\log_{10} x)^2 - \log x - 6 = 0$.

12. $\dfrac{\log_{10} (x + 1)}{\log_{10} x} = 2$. *Hint:* $2 \log_{10} x = \log_{10} x^2$.

13. $\log_{10} x + \log_{10} (x + 3) = 1$.

14. $12(\log_{10} x)^2 + 5 \log x - 2 = 0$.

15. $1 + \log_{10} x = \log_{10} (1 + x)$.

16. In a geometric progression, $l = ar^{n-1}$, solve for n in terms of a, l, and r.

17. In a geometric progression, $s = \dfrac{ar^n - a}{r - 1}$, solve for n in terms of a, r, and s.

Solve for x and y the following systems of equations:

18.
$$5^{x+y} = 82, \qquad (1)$$
$$3^{x-y} = 4. \qquad (2)$$

Solution: From (1) and (2),

$$(x + y) \log 5 = \log 82, \qquad (3)$$
$$(x - y) \log 3 = \log 4. \qquad (4)$$

Solving the linear equations (3) and (4) for x and y, we get

$$x = \frac{\log 82}{2 \log 5} + \frac{\log 4}{2 \log 3} = \frac{1.9138}{1.398} + \frac{0.6021}{0.9542} = 2.000. \qquad (5)$$

$$y = \frac{\log 82}{2 \log 5} - \frac{\log 4}{2 \log 4} = 0.7380. \qquad (6)$$

19. $3^{x+y} = 10,$
$2^x = 3^5.$

20. $5^{2x+y} = 7^{3x},$
$x - y + 5 = 0.$

21. Solve for x the equation $e^x + e^{-x} = y$; (*a*) when $y = 2$, (*b*) when $y = 4$.

22. If fluid friction be used to retard the motion of a flywheel making V_0 revolutions per minute, the formula $V = V_0e^{-kt}$ gives the number of revolutions per minute, after the friction has been applied t seconds. If the constant $k = 0.35$, how long must the friction be applied to reduce the number of revolutions from 500 to 50 per minute?

23. The pressure, P, of the atmosphere in pounds per square inch, at a height of z feet, is given approximately by the relation

$$P = P_0e^{-kz},$$

where P_0 is the pressure at sea level and k is a constant. Observations at sea level give $P_0 = 14.72$, and at a height of 1122 feet, $P = 14.11$. What is the value of k?

24. Assuming the law in problem 23 to hold, at what height will the pressure be half as great as at sea level?

25. If a body of temperature $T_1°$ be surrounded by cooler air of temperature $T_0°$, the body will gradually become cooler and its temperature, $T°$, after a certain time, say t minutes, is given by Newton's law of cooling, that is,

$$T = T_0 + (T_1 - T_0)e^{-kt},$$

where k is a constant. In an experiment a body of temperature 55° C. was left to itself in air whose temperature was 15° C. After 12 minutes the temperature was found to be 23.8°. What is the value of k?

26. Assuming the value of k found in problem 25, what time will elapse before the temperature of the body drops from 23.8 to 20°?

27. If $a = \log_c b$, $b = \log_a c$, $c = \log_b a$, prove that $a \cdot b \cdot c = 1$.

28. In solving an important problem in the elements of mechanics, it turns out that

$$t = \frac{1}{k} \log_e \frac{ks + \sqrt{k^2s^2 + v_0^2}}{v_0},$$

where s is the distance traversed by a moving point in time t. It is, in general, more useful to have s in terms of t than t in terms of s. Hence, express s in terms of t.

147. Calculation of logarithms. At this point the inquiring student will naturally bring up the question as to how the logarithms of numbers are computed so as to make a table of logarithms. Logarithms were invented by Napier about the year 1600 and common logarithms by Briggs a little later. The invention grew out of the comparison of two series of numbers — the one in arithmetic progression and the other in geometric progression. The following theorem lies at the foundation of the early methods of computing logarithms:

If a series of numbers are in geometric progression, their corresponding logarithms are in arithmetic progression.

Let the numbers in geometric progression be

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}. \quad (1)$$

Then, $$\log a, \log ar, \log ar^2, \log ar^3, \cdots, \log ar^{n-1} \quad (2)$$

are in arithmetic progression.

In this arithmetic progression, the first term is $\log a$, and the common difference is $\log r$. The following example illlustrates the use of this principle in calculating logarithms:

Given $\log 1 = 0$, $\log_{10} 1000 = 3$, the geometric mean between 1 and 1000 is $\sqrt{1000} = 31.62$. Then 1, 31.62, 1000 is a geometric progression, and 0, 1.5, 3 is the corresponding arithmetic progression, so that $1.5 = \log_{10} 31.62$. Next, insert a geometric mean between 1 and 31.62, also between 31.62 and 1000. This gives

1, 5.624, 31.62, 177.8, 1000 as the geometric series,

and 0, 0.75, 1.5, 2.25, 3 as the corresponding logarithms.

We could next insert between any two of these numbers a geometric mean, and find its logarithm. By continuing this process, we could insert means until the numbers would differ by as little as we please. This method of calculating logarithms has the disadvantage of giving the logarithms of numbers spaced unequally, since the numbers are in geometric progression.

Another method of obtaining logarithms, which has many advantages over the one just given, is discussed briefly in Art. 201 of the chapter on Infinite Series; but its more complete treatment belongs to the calculus.

CHAPTER XVI

COMPOUND INTEREST AND ANNUITIES

148. Compound interest. When interest due is added to the principal at stated intervals, say annually, semiannually, or quarterly, the interest is said to be **compounded** or **converted** into principal. The **conversion period** is the time between two successive conversions of interest into principal. If no conversion period is named, it is ordinarily understood to be one year.

If P is the original principal, i the rate per conversion period, the sum S, called the **compound amount,** to which P will accumulate at the end of n conversion periods is

$$S = P(1 + i)^n. \tag{1}$$

To prove this we may proceed as follows:

Interest due at the end of the first period is Pi. The amount at the end of the first period is

$$P + Pi = P(1 + i).$$

The new principal after the first conversion period is thus obtained by multiplying the principal at the beginning of the period by $1 + i$. Applying this multiplier to the successive principals, for n periods, we obtain the amount S given by (1).

The compound amount diminished by the original principal is called the **compound interest.**

149. Present value. The problem of finding the **present value** of an amount S due after n years at rate i per year is solved by finding P in (1) when S, i, and n are given. That is, the present value is

$$P = \frac{S}{(1 + i)^n}. \tag{2}$$

In the absence of compound interest tables,* the computations involving compound interest offer applications for practice in the use of logarithms.

* Tables of compound interest and annuities certain, sufficiently extensive for many practical purposes, are given in books on mathematics of finance. For a set of such tables, see *Mathematics of Finance*, by Rietz, Crathorne, and Rietz, Henry Holt and Company.

EXERCISES AND PROBLEMS

1. Find the amount of \$1000 in 10 years at 3% interest compounded annually.

In this case, $S = \$1000\ (1.03)^{10}$.

Form for solution by logarithms:

$$\begin{array}{r l}
\log 1.03 & = \underline{\qquad\qquad} \\
\log (1.03)^{10} & = \\
\log 1000 & = \underline{\qquad\qquad} \\
\log S & = \\
S & =
\end{array}$$

2. What principal will amount to \$2000 in 10 years at the rate .03 compounded annually?

Form for solution: $$P = \frac{\$2000}{(1.03)^{10}}.$$

$$\begin{array}{r l}
\log 1.03 & = \underline{\qquad\qquad} \\
\log 2000 & = \\
\log (1.03)^{10} & = \underline{\qquad\qquad} \\
\log P & = \\
P & =
\end{array}$$

3. Find the amount of \$1000 in 10 years at 3%, interest converted semiannually.

Hint: $S = \$1000\ (1.015)^{20}$.

Find the compound amount and the compound interest, to the nearest dollar.

4. Of \$600 for 8 years at 4%, compounded annually.

5. Of \$180 for 4 years at 5%, compounded semiannually.

6. Of \$900 for $4\frac{1}{2}$ years at 6%, compounded semiannually.

7. What sum of money invested at the rate of .03 compounded semiannually from a child's birth will give him \$2000 at age 21?

8. In how many years will any sum double itself at the rate .04 compounded annually?

9. In how many years would \$100 amount to \$400 at the rate .06, converted quarterly?

10. Construct a graph of the function $y = (1.04)^x$ to show the variation of the amount y with respect to the time x.

11. If \$1 had been kept on interest at 3%, compounded annually, from the beginning of the Christian Era to the present time, how many digits would occur in the integral part of the accumulated amount when expressed in dollars?

12. If the \$24 said to have been paid to the Indians in 1626 by Peter Minuit for Manhattan Island had been on interest at 4% compounded annually until 1939, find the accumulated amount expressed in dollars correct to two significant figures.

13. If interest is at nominal rate j per year but is converted m times a year, show that formula (1), Art. 148 becomes

$$S = P\left(1 + \frac{j}{m}\right)^{mt},$$

where t is the time in years. Check your solution of problem 3 on page 204 by the use of this formula.

14. If the number m of interest conversions per year in problem 13 increases beyond bound, the interest is said to be converted continuously and the formula for the amount becomes

$$S = Pe^{jt},$$

where e is the base of natural logarithms. (See Arts. 144 and 146.) Find the amount of $100 in 5 years at .04 converted continuously.

15. Find the amount after 20 years if $1000 is invested at the rate 5%:

(*a*) converted annually.
(*b*) converted semiannually.
(*c*) converted continuously.

150. Annuities certain. An **annuity certain** is a set of equal payments made at equal intervals over a fixed period of time. For example, suppose a fraternity is in debt on a chapter house, and is to pay $2500 at the end of each year for 20 years to discharge the debt, both interest and principal. The set of payments constitute an annuity certain.

Two questions about annuities naturally arise:

(1) To what amount would the payments accumulate at the end of the paying period?

(2) What is the present value of the payments?

For simplicity, we shall limit* our answers to these questions to cases in which the payments are made yearly with the first payments at the end of the year, and the rate is i compounded annually. To take up the problem involved in question (1) let R be the yearly payment.

The first payment of R will accumulate to	$R(1 + i)^{n-1}$.
The second payment of R will accumulate to	$R(1 + i)^{n-2}$.
The third payment of R will accumulate to	$R(1 + i)^{n-3}$.
The next to the last payment of R will accumulate to	$R(1 + i)$.
The last payment of R will accumulate to	R.

* For a more complete treatment, see *Mathematics of Finance*, by Rietz, Crathorne, and Rietz, Henry Holt and Company.

Reversing the order and adding, and letting K be the sum we have

$$K = R[1 + (1 + i) + \cdots + (1 + i)^{n-2} + (1 + i)^{n-1}].$$

The right-hand member is a geometric progression of n terms in which the first term is R and the common ratio is $1 + i$. The sum, called the **amount of the annuity,** is then (Art. 84)

$$K = R\frac{(1 + i)^n - 1}{i}. \tag{3}$$

To take up the problem involved in question (2), we define the **present value of an annuity** as the sum of the present values (Art. 149) of the separate payments. The present values of the payments of R each beginning with the first one are

$$R(1 + i)^{-1}, R(1 + i)^{-2}, \cdots, R(1 + i)^{-n}.$$

We have then for the present value of the annuity

$$P = R[(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-n}],$$

a geometric progression whose first term is $R(1 + i)^{-1}$, last term $R(1 + i)^{-n}$, and common ratio $(1 + i)^{-1}$. Summing the series we have (Art. 84)

$$P = R\frac{1 - (1 + i)^{-n}}{i}. \tag{4}$$

EXERCISES AND PROBLEMS

1. A man sets aside \$200 at the end of each year towards a fund for his son's college expenses. He invests the money at rate .04 compounded annually. Calculate the amount at the end of 15 years as nearly as you can using a five-place table of logarithms to find $(1.04)^{15}$.

Solution: From (3) we have the amount

$$S = 200\frac{(1.04)^{15} - 1}{.04}.$$

By five-place logarithms $(1.04)^{15}$ is found to be 1.801.
Hence $(1.04)^{15} - 1 = .801$.

$$S = \frac{200}{0.4}(.801) = \$4005,$$

correct somewhat accidentally to the nearest dollar.

2. A man pays \$26.08 paving tax at the end of each year for 10 years. If the interest charge is 5%, what is the actual tax for the paving to the nearest dollar?

Solution: From (4), we have for the present value of the annuity certain

$$P = 26.08\frac{1-(1.05)^{-10}}{.05}.$$

By four-place logarithms, $(1.05)^{-10} = .6137$, hence $1 - (1.05)^{-10} = .3863$.

$$\text{Hence, } P = \frac{(26.08)(.3863)}{.05} = \$201.$$

Find the amount and present value of the annuity described, correct to the nearest dollar.

3. \$300 at the end of each year for 10 years, at 3% compounded annually

4. \$150 at the end of each half-year for 20 years, at 4% per annum compounded semiannually.

5. To discharge a debt on a house and lot the owner agrees to pay \$1000 at the end of each year for the next eight years. What is the equivalent cash price to the nearest dollar if money is worth 5% convertible annually?

6. The beneficiary of a life insurance policy is to receive \$1000 at the death of the insured, and \$1000 at the end of each of the next nine years. Find to the nearest dollar the equivalent cash payment at the date of death of the insured if money is worth $3\frac{1}{2}\%$ converted annually.

7. The amount of an annuity of \$750 per annum for three years is \$2341.20. Write down an equation whose solution for i is the rate of interest. Show by substitution that .04 satisfies the equation.

8. Test the accuracy with which you can compute the amount of an annuity of \$1000 per year for 10 years at rate .06 converted annually,

(*a*) by using four-place logarithms,
(*b*) by using five-place logarithms,
(*c*) by using seven-place logarithms.

The answer to be used as a test is \$13,180.79 to the nearest cent.

9. A debt of \$8000, at 6% compounded annually, is discharged by eight equal annual payments at the ends of the years. Required the annual payment to the nearest dollar.

10. A house for sale is listed at \$8000. The seller agrees to take \$3200 cash and \$800 per annum for 6 years without interest. If money is worth 6% per annum in such transactions, what reduction was made in the price of the house?

CHAPTER XVII

PERMUTATIONS AND COMBINATIONS

151. Introduction. Two positions are to be filled in an office — one that of stenographer and the other that of messenger. There are 12 applicants for the position of stenographer, and 3 for that of messenger. In how many ways can the two positions together be filled?

The position of stenographer can be filled in 12 ways, and with each of these there is a choice of 3 messengers. Hence, the two positions can be filled in $12 \times 3 = 36$ ways.

This example illustrates the following

FUNDAMENTAL PRINCIPLE. *If one thing can be done in m different ways; and if, after this is done in one of these ways, a second thing can be done in n ways, then the two together can be done in the order stated in mn ways.*

For, corresponding to each of m ways of doing the first thing, there are n ways of doing the second thing. In other words, there are n ways of doing the two together for each way of doing the first thing. Hence, there are in all mn ways of doing the two things together.

A convenient and evident extension of the fundamental principle may be stated in the following form:

If one thing can be done in m_1 ways, a second in m_2 ways, a third in m_3 ways, and so on, the number of different ways in which they can be done when taken all together in the order stated is $m_1m_2m_3 \cdots$.

152. Meaning of a permutation. *Each different arrangement which can be made of all or part of a number of things is called a* ***permutation.***

By the expression "number of permutations of n things taken r at a time" is meant the number of permutations consisting of r things each which can be formed from n different things. Thus, the permutations of the letters abc taken all at a time are

$$a\,b\,c, \quad a\,c\,b, \quad b\,a\,c, \quad b\,c\,a, \quad c\,a\,b, \quad c\,b\,a$$

The permutations of the four letters $a\ b\ c\ d$ taken three at a time are

$$\begin{array}{cccc} abc & bac & cab & dab \\ acb & bca & cba & dba \\ acd & bcd & cbd & dbc \\ adc & bdc & cdb & dcb \\ abd & bad & cad & dac \\ adb & bda & cda & dca \end{array}$$

153. Permutations of things all different. The special cases just considered lead us to the problem of deriving a formula for the number of permutations of n things taken r at a time. The symbol $P(n, r)$ is used to represent this number.

The number of permutations of n different things taken r at a time is

$$P(n, r) = n(n-1) \cdots (n-r+1).$$

The number $P(n, r)$ required is the same as the number of ways of filling r different positions with n different things. We may represent the n things by $a_1, a_2, \cdots, a_n$ and ask how many permutations of r letters can be formed from them. For the first place there is a choice of n letters, for the second a choice of $n - 1$, for the third a choice of $n - 2$, and so on. For the rth place there is then a choice of $n - r + 1$ letters. It follows (Art. 151) that

$$P(n, r) = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!} \qquad (1)$$

When $r = n$, (1) becomes

$$P(n, n) = n(n-1) \cdots 2 \cdot 1 = n!. \qquad (2)$$

That is, the number of permutations of n things taken n at a time is $n!$.

154. Permutations of n things not all different. Consider the number of permutations of the letters in the word *book*. It gives no new permutation to interchange the *o*'s. Let P be the number of permutations. If we should replace *oo* by dissimilar characters o_1o_2, there would be $2!$ permutations of o_1o_2 corresponding to each of the P permutations. But if the letters were all different the number of permutations would be $4!$. Hence,

$$4! = 2!\,P, \quad P = \frac{4!}{2} = 12.$$

This example illustrates the

THEOREM. *If P is the number of permutations of n things taken all at a time, of which n_1 are alike, n_2 others alike, n_3 others alike, and so on, then*

$$P = \frac{n!}{n_1!\, n_2!\, n_3! \cdots}.$$

To establish the theorem, suppose we should replace n_1 like things by n_1 unlike things, there would be $P \cdot n_1!$ permutations obtained from the original P permutations. In each of these permutations there would be n_2 things alike, and n_3 others alike. Similarly, replacing the n_2 like things by n_2 dissimilar things, we get $P \cdot n_1! \cdot n_2!$ permutations in each of which there would be n_3 alike. Continuing this argument, we find that the number of permutations of n things taken all at a time, when n_1 are alike, n_2 others alike, n_3 others alike, and so on, is given by

$$P = \frac{n!}{n_1!\, n_2!\, n_3! \cdots}.$$

EXERCISES AND PROBLEMS

1. How many different permutations can be made of the letters of the word "numbers" when taken three at a time; four at a time?

2. Five cards each having a different design may be arranged in a row in how many different ways?

3. If two sets of the cards in problem 2 are shuffled together and spread out in a row, how many arrangements may be made?

4. If the cards in problem 2 were placed equally spaced about a ring, how many different arrangements could be made?

5. If the cards in problem 3 were placed equally spaced about a ring, how many different arrangements could be made?

6. A certain automobile is made in five body types, three choices of wheels, and six different color schemes. How many cars are necessary for an exhibit showing all possible cars?

7. In how many ways can nine books be arranged on a shelf if two of the books are to be kept together?

8. Given $P(n, 7) = 30P(n, 5)$, find n.

9. What would be the maximum number of Greek letter fraternities having distinct names consisting of three different letters (the Greek alphabet contains 24 letters)?

10. How many Greek letter fraternities may be organized having names of three letters, repetitions of the letters being allowed?

11. How many Greek letter fraternities are possible having either two or three letters in their names?

12. If all the letters of each of the words in "college algebra" are rearranged, how many pairs of permutations are possible?

13. Solve for n, $3P(n, 5) = 68P(n - 2, 4)$.

14. How many integers less than a million contain the digits 1, 2, 3, 4 in the order given?

155. Combinations. *A set of things or elements without reference to the order of individuals within the set is called a* **combination.**

Thus *abc*, *acb*, *bac*, *bca*, *cab*, *cba* are the same combination. By the "number of combinations of n things taken r at a time" is meant the number of combinations of r individuals which can be formed from n things.

Thus, the combinations of $a\ b\ c$ taken two at a time are *ab*, *ac*, *bc*.

156. Combinations of things all different. Let $C(n, r)$ denote the number of combinations of n things taken r at a time. Then a formula can be derived for $C(n, r)$ by establishing the relation between $C(n, r)$ and $P(n, r)$.

Take one combination of r things; with this $r!$ permutations can be made. Take a second combination; with this $r!$ permutations can be made. There are thus $r!$ permutations for each combination. Hence, there are in all $C(n, r) \cdot r!$ permutations of n things taken r at a time. That is,

$$C(n, r) \cdot r! = P(n, r)$$

whence $$C(n, r) = \frac{P(n, r)}{r!}.$$

Since $$P(n, r) = n(n - 1) \cdots (n - r + 1), \qquad \text{(Art. 153)}$$

we have $$\boldsymbol{C(n, r) = \frac{n(n - 1) \cdots (n - r + 1)}{r!}}.$$

Multiplying numerator and denominator by $(n - r)!$, we get

$$\boldsymbol{C(n, r) = \frac{n!}{r!(n - r)!}}.$$

Since $$C(n, n - r) = \frac{n(n - 1) \cdots (r + 1)}{(n - r)!} = \frac{n!}{(n - r)!r!},$$

it follows that the number of combinations of n things taken r at a time is the same as the number taken $n - r$ at a time.

157. Binomial coefficients. It may be noted that the formula for $C(n, r)$ is the coefficient of the $(r + 1)$st term of the binomial expansion $(a + x)^n$. The binomial theorem for positive integral exponents may therefore be written in the form

$$(a+x)^n = a^n + C(n, 1)a^{n-1}x + C(n, 2)a^{n-2}x^2 + \cdots + C(n, n-1)ax^{n-1} + C(n, n)x^n.$$

158. Total number of combinations. *The total number of combinations of n things taken* $1, 2, 3, \cdots, n$ *at a time is* $2^n - 1$. If we write the binomial theorem as in the last section, we obtain

$$(1+x)^n = 1 + C(n, 1)x + C(n, 2)x^2 + \cdots + C(n, n-1)x^{n-1} + C(n, n)x^n.$$

Putting $x = 1$, we get

$$2^n - 1 = C(n, 1) + C(n, 2) + \cdots + C(n, n-1) + C(n, n).$$

EXERCISES AND PROBLEMS

1. How many different debating teams of three men each may be selected from eight candidates?

2. How many distinct straight lines may be drawn through seven points no three of which are on the same straight line?

3. From four employers and eight employees in how many ways can a committee of five be chosen to include one and only one employer?

4. If any arrangement of letters is considered to be a word how many four letter words are possible from the English alphabet (*a*) if each word must contain at least one vowel; (*b*) if no repetitions of any letter are allowed; (*c*) if repetitions are allowed?

5. In an examination with 13 questions the student is to answer 10, two or three of which must be chosen from the first three questions. In how many ways may the student choose the ten questions?

6. Prove that $C(n, r) = C(n, n - r)$.

7. Given $C(n, 2) = 153$. Find n.

8. Given $P(n, r) = 840$, $C(n, r) = 35$. Find n and r.

9. Solve for n, $P(n, 4) = 30\,C(n - 1, 3)$.

10. Solve for n, $35C(2n, n - 1) = 132C(2n - 2, n)$.

11. Solve for r, $\dfrac{C(7, r)}{C(9, r)} = \dfrac{5}{18}$.

12. In how many ways can a pack of 52 playing cards be divided into four hands, the order of the hands, but not the cards in the hands, to be regarded?

13. How many committees each consisting of five students and two professors may be formed from a group of ten students and four professors?

14. In how many ways may six books be arranged on a shelf so that no two of three particular books are ever together?

15. Find the number of ways of dividing 11 things into groups of 5 and 6.

16. Find the number of ways of dividing 12 things into two equal groups.

17. Show that the number of ways of dividing $2n$ things into two equal groups is the same as the number of ways of dividing $2n - 1$ things into groups of n and $n - 1$ things.

18. In how many ways may three different prizes be given to three boys when each is eligible for all the prizes?

19. How many different products can be formed from the five numbers 2, 3, 5, 7, 11 taking two or more numbers at a time?

20. How many different products can be formed from the five numbers 2, 3, 4, 5, 6 taking two or more numbers at a time?

21. In how many ways is it possible to draw a sum of money from a bag containing a dollar, a half dollar, a quarter, a dime, a nickel, and a cent.

22. In how many ways can a bridge hand of 13 cards be made up of 3 diamonds, 3 hearts, 3 spades and 4 clubs?

23. If $K = C(n, 2)$, show that $C(K, 2)$ is three times $C(n + 1, 4)$.

24. How many different combinations can be formed with the following weights?

1 1-gram	1 10-gram	1 100-gram
1 2-gram	1 20-gram	1 200-gram
1 3-gram	1 30-gram	1 300-gram
1 5-gram	1 50-gram	1 500-gram
		1 1000-gram

25. Find an expression for the number of permutations of n things taken three at a time when two of the n things are alike.

26. Find an expression for the number of combinations of n things taken three at a time when two of the n things are alike.

27. How many baseball teams of 9 men each may be chosen from 14 players of whom 7 are qualified to play in the infield only, 5 in the outfield only and 2 in any position? The battery is included in the infield.

CHAPTER XVIII

RELATIVE FREQUENCY AND PROBABILITY

159. Meaning of relative frequency. A bag contains white and black balls alike except as to color and thoroughly mixed. The drawing of a ball and replacing it is called a **trial.** Suppose we make 100 trials and obtain 31 white balls. Then we say $\frac{31}{100}$ is the relative frequency of white balls in this set of drawings.

In making a trial we often call the happening of the event in question a **success.** In the above case the drawing of a white ball may be called a success and the drawing of a black ball a **failure.** In general, if we make n trials resulting in m successes and $n - m$ failures, we say that $\frac{m}{n}$ is the relative frequency of successes and $\frac{n-m}{n}$ is the relative frequency of failures in the n trials.

The sum of the relative frequencies of successes and of failures is clearly equal to $\frac{m}{n} + \frac{n-m}{n} = 1.$

Query. What was the relative frequency of deaths in a year among 10,000 persons of equal age if there were 50 deaths within a year?

160. Meaning of probability of success. If we increase the number of trials described in Art. 159 from 100 to any larger number, say to 1000 or more, we would not necessarily find exactly the same relative frequency of successes. Next, if we conceive of increasing the number of trials and of calculating a new value of the relative frequency of successes after each trial we may find that the sequence of relative frequencies approaches a limiting value (See Art. 181). If there is such a limiting value, it is called the **probability of success in one trial.** Thus, if we conceive of repeating indefinitely the drawing of a ball from a thorough mixture of balls, three-tenths of which are white, we may assume that the relative frequency of white balls would approach three-tenths and we say three-tenths is the probability of obtaining a white ball in one trial. This illustrates the following definition of the probability of success in one trial:

If the relative frequency of successes approaches a limit when the trial is repeated indefinitely under the same set of circumstances, this limit is called the probability of success in one trial.

Since the sum of the relative frequency of successes and of failures is 1, it readily follows that *the sum of the probability of success and of failure is* 1.

In framing this definition we idealize our actual experience. We say the probability that a penny will fall "heads up" is one half. This may be looked upon as an answer to the following question: What should we expect in the long run for the ratio of the number of heads to the total number of pennies tossed?

When applying the above definition of probability, a question very naturally arises about the meaning of the expression, "the same set of circumstances"; for we may in a sense question whether two or more dice could be tossed under the same circumstances, or again whether two or more men of the same age could live under the same set of circumstances. Without taking the space to discuss at length this question, some light may be thrown on it by saying that the expression implies the absence of specific differences. For example, since we can specify the difference between loaded dice and unloaded dice, between healthy and diseased men, between old and young men, we do not include two such kinds in the same group.

161. Approximate probability derived from observation. *After we have obtained a relative frequency of successes m/n in n trials (n a large number); then in the absence of further knowledge, it is usually assumed that $\frac{m}{n}$ is a good estimate of probability of success in a given trial and that confidence in this estimate increases as n increases.*

Such estimates of probability are of much practical value in insurance and statistics. For example, if we observe 80,000 men of a well defined class, say of age 30, and find that 480 deaths occurred during the year, we give, $\frac{480}{80{,}000} = .006$ as an estimate of probability that a man of this class will die within a year.

162. Probability derived from an analysis into equally likely ways. In certain cases, notably in games of chance, the probability may be obtained by an analysis of all trials into a certain number of equally likely ways. For example, consider the case of a bag containing three white and seven black balls. What is the probability that a ball to be drawn will be white? To answer this question, we may analyze all possible drawings into 10 equally likely cases of which three will give white balls. We give 3/10 as the probability of drawing a white ball. This simple case illustrates the following process of arriving at a probability:

If all the successes and failures can be analyzed into $r + s$ possible ways each of which is equally likely; and if r of these ways give successes, and s of them failures, the probability of success in a single trial is $\frac{r}{r+s}$ and the probability of failure in the trial is $\frac{s}{r+s}$.

In this connection, the fact should not be overlooked that the ways were assumed to be "equally likely." To illustrate the need of precaution in this matter, consider the following

Example: What is the probability that a man, A, in good health will die within the next 24 hours?

We might argue that the event can happen in only one way and fail in only one way, and that the probability that A will die in the next 24 hours is therefore $\frac{1}{2}$. What is the flaw in this argument?

The expression "equally likely" indicates that we have no more reason to expect the event to take place in one way than in any other.

In the above analysis into equally likely ways, the **odds** are said to be r to s in favor of the event if $r > s$, r to s against it if $r < s$, **even** if $r = s$.

ORAL EXERCISES

1. What is the probability that a coin tossed at random will fall "heads up"?

2. What is the probability of obtaining an ace in throwing a single die?

3. If the probability of losing a game is 0.55, what is the probability of winning it?

4. From a class of 25 students of whom 10 are girls, one student is to be selected by lot. What is the probability that a girl will be selected?

5. Out of 40 children born in a village in a given year, 22 were boys and 18 were girls. What is the relative frequency of girls among the children born in the village in the year?

6. The odds are even that A will win a game. What is the probability that he will win it?

163. Mathematical expectation of money. If p is the probability that a person will win a sum of money m, we may define his **mathematical expectation** as pm.

164. Expected number of occurrences. The **expected number of occurrences** of an event in n trials is defined to be np, where p is the probability of occurrence of the event in a single trial. It is an immediate and useful consequence of this definition that the probability of occurrence of an event is the ratio, $\frac{np}{n} = p$, of the expected number of occurrences to the number of trials.

PROBLEMS

1. If a man makes a single trial on a gambling machine where the stake is \$30 and where the probability of winning in one trial is $\frac{9}{20}$, what is his mathematical expectation?

2. In a lottery, the prize is \$10, and 100 tickets have been issued. What is the mathematical expectation of a man with 15 tickets?

3. A bag contains 4 white balls, 3 red balls, and 2 black balls. What is the probability that a ball drawn at random will be white? Will be red? Black? Red or black?

4. If the odds are 3 to 2 in favor of a man winning a prize of \$100, find (*a*) his probability of winning, and (*b*) his mathematical expectation.

5. In the United States in 1933 there were 2,081,232 live births. Of these children 1,068,871 were boys and 1,012,361 were girls. Compute the relative frequency of boys among the children born in the United States in 1933, correct to four significant figures.

6. It is suggested that each student in the class throw a coin at random 50 times and record the number of heads. Combine the results for the class and find the relative frequency of heads. Compare the result with the probability, $\frac{1}{2}$, of throwing a head with a single coin.

7. Six coins are tossed. What is the probability that exactly two of them are heads?

Solution: Since each coin can fall in two ways, the six can fall in $2^6 = 64$ ways. The two coins can be selected from the six in $C(6, 2) = 15$ ways. Hence the probability is $\frac{15}{64}$.

8. From a bag containing 8 white and 4 black balls, 2 are drawn at random. Find the probability that (*a*) both are white; (*b*) one is white and one is black.

9. From 10 men and 5 women, a committee of 4 is chosen by lot. Find the probability that the committee will consist of 2 men and 2 women.

10. At a bridge party attended by four married couples, each man is to be assigned a woman partner by lot. Find the probability that each man will draw his own wife as a partner.

11. A gambler is to win \$300 if an ace is thrown with a single die. What is his mathematical expectation?

12. From a suit of 13 spades, 3 cards are to be drawn. What is the probability that an ace, a king, and a queen will be drawn?

13. Volumes I, II, III, IV of Cantor's *History of Mathematics* are placed on a library shelf at random. What is the probability that the volumes will be in the correct order: I, II, III, IV?

14. If 0.006 be the probability of death within a year of a man aged 35, what would be the expected number of deaths within a year among 25,000 such men?

15. According to the so-called "American Experience Table of Mortality" (constructed in 1868 when relatively little experience was available on insured lives in America), out of 89,032 persons living at age 25, there are 69,804 who reach age 50. From these data, estimate, correct to three significant figures, the probability that a man aged 25 would not live to reach age 50.

16. According to the "American-Men Mortality Table" (constructed from an immense number of insured lives in the United States and Canada exposed to risk between January 1, 1900 and January 1, 1915), out of 96,203 men living at age 25, there are 82,805 who reach age 50. From these data, estimate, correct to three significant figures, the probability that a man aged 25 would not live to reach age 50. Compare this result with that derived from the older table in problem 15.

165. Theorems of total and compound probability. Consider the questions: What is the probability of throwing either an ace or a deuce in a single throw with a die? What is the probability of throwing two aces in a single throw with two dice? The first question belongs to a class of questions answered by applications of a proposition called the theorem of total probability, the second to a class answered by applications of a proposition called the theorem of compound probability.

Let E_1, E_2, $\cdots$, E_r be a set of r events whose probabilities of occurrence in any single trial are p_1, p_2, $\cdots$, p_r respectively. The expected number of occurrences (Art. 164) of the several events in n trials is np_1, np_2, $\cdots$, np_r respectively.

Exclusive events. The events of a set are said to be **mutually exclusive** when the occurrence of any one of them in a trial excludes the occurrence of any other in that trial. Thus, the throwing of an ace and a deuce with a single die are mutually exclusive events.

When the events $E_1, E_2, \cdots, E_r$ are mutually exclusive, the expected number of occurrences for the total set of r events in n trials is

$$np_1 + np_2 + \cdots + np_r, \tag{1}$$

the sum of the expected values for the separate events.

Since the total probability P for an occurrence is the ratio of the expected number of occurrences (Art. 164) for the whole set to the number of trials, we have

$$P = \frac{np_1 + np_2 + \cdots + np_r}{n} = p_1 + p_2 + \cdots + p_r, \tag{2}$$

which may be stated as the

Theorem of total probability. *The probability that some one or other of a set of mutually exclusive events will happen in a single trial is the sum of the probabilities for the separate events.*

Thus, the probability of throwing either an ace or a deuce with one die is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Independent events. The events of a set are said to be **mutually independent** or **dependent** according as the occurrence of one of them does not or does affect the probability of occurrence of others in the set.

In n trials, where n is a large number, the expected number of occurrences of event E_1 is np_1 (Art. 164). Out of this number, np_1, the expected number of occurrences of event E_2 is $p_2(np_1) = np_1p_2$. That is, both are expected to occur np_1p_2 times in the n trials. Continuing this process, the expected number of occurrences of all of the r events is

$$np_1p_2 \cdots p_r.$$

Then from Art. 164, the probability, P, that all of the r events will happen in one trial is the ratio

$$P = \frac{np_1p_2 \cdots p_r}{n} = p_1p_2, \cdots p_r,$$

which may be stated as the

Theorem of compound probability for independent events. *The probability that all of a set of independent events will occur on a given occasion when all of them are in question is the product of their separate probabilities.*

Thus, the probability of throwing two aces in a single throw with two dice is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

Theorem for dependent events. *If the probability of a first event is p_1, and if, after this has happened, the probability of a second event is p_2; then the probability that both events will happen in the order stated is p_1p_2. The extension to any number of events is obvious.*

PROBLEMS

1. What is the probability of getting either two heads or two tails in throwing two coins?

2. A bag contains 10 white, 5 red, and 3 black balls. A ball is to be drawn. What is the probability that it is either white or red?

3. The probability that A will win a game is $\frac{1}{3}$, and that B will win another independent game is $\frac{1}{4}$. What is the probability that both will win?

4. The probability that A will live ten years is $\frac{3}{4}$ and that B will live ten years is $\frac{4}{5}$. What is the probability that both will live ten years?

5. What is the probability of throwing either an ace, or a deuce, or a trey with a single die?

6. Find the probability of drawing 2 white balls in succession from a bag containing 5 white and 6 black balls if the first ball drawn is not replaced before the second drawing is made.

7. One purse contains 9 coins consisting of 2 dimes, 3 quarters, and 4 half dollars. If one coin is drawn at random from the purse, what is the probability of its being either a quarter or a half dollar?

8. If the probability is $\frac{1}{6}$ that the height of a man selected at random from a group of men is between 5 feet 8 inches and 5 feet 9 inches, and $\frac{1}{8}$ that it is between 5 feet 7 inches and 5 feet 8 inches, what is the probability that his height is between 5 feet 7 inches and 5 feet 9 inches?

9. A bag contains 6 balls marked 1, 2, 3, 4, 5, 6. A ball is drawn and not replaced. A second ball is drawn from the 5 remaining in the bag. What is the probability that the first ball drawn was marked 1 and the second 2?

10. A coin is thrown twice. What is the probability that the first throw gave a head and the second throw a tail?

11. A traveler has three independent railroad connections to make. If the probability is $\frac{2}{3}$ that he would make any particular connection taken alone, what is the probability of his making all three connections?

12. The probability that a man of a certain age will die within 20 years is 0.2, and that his wife will die within that time is 0.15. What is the probability that at the end of 20 years (*a*) both will be dead? (*b*) both will be living? (*c*) the man will be living and his wife dead? (*d*) the man will be dead and his wife living?

166. Repeated trials. *If p is the probability that an event will happen in any single trial, then $C(n, r)p^rq^{n-r}$ is the probability that this event will happen exactly r times in n trials, where $q = 1 - p$ is the probability that the event will fail in any single trial.*

For, the probability that it will happen in each of r specified trials and fail in all the remaining $n - r$ trials is p^rq^{n-r} (Art. 165), and r trials can be selected from n trials in $C(n, r)$ ways. These ways being mutually exclusive, we have by Art. 165, that the probability in question is

$$C(n, r)p^rq^{n-r}.$$

It will be observed that $C(n, r)p^rq^{n-r}$ is the $(n - r + 1)$th term of the binomial expansion of $(p + q)^n$.

Illustration: Three dice are to be thrown. What is the probability of obtaining exactly two aces?

Solution: The probability given by the second term of the binomial expansion $\left(\frac{1}{6} + \frac{5}{6}\right)^3$ is $\frac{5}{72}$.

We next inquire into the probability that an event such as is described above happens at least r times in n trials. The event happens *at least* r times if it happens exactly $n, n - 1, n - 2, \cdots$, or r times in n trials.

Hence, we have the following

THEOREM. *The probability that an event will happen at least r times in n trials is $p^n + C(n, n - 1)p^{n-1}q + C(n, n - 2)p^{n-2}q^2 + \cdots + C(n, r)p^rq^{n-r}$.*

This expression is the first $n - r + 1$ terms of the binomial expansion of $(p + q)^n$.

Illustration: Three dice are to be thrown. What is the probability of obtaining at least two aces?

Solution: The probability given by the sum of the first and second terms of the expansion $\left(\frac{1}{6} + \frac{5}{6}\right)^3$ is $\frac{2}{27}$.

PROBLEMS

1. Four coins are to be thrown. What is the probability that (*a*) exactly two heads will fall up, (*b*) at least two heads?

2. In throwing 3 dice, what is the probability of no aces?

3. In tossing 10 coins, what is the probability that (*a*) just 4 of them will be heads, (*b*) at least 4 will be heads?

4. A's chance (probability) of winning any single game against B is $\frac{3}{4}$. Find the chance of his winning at least three games out of seven.

5. If, in the long run, one vessel out of every 50 is wrecked, find the probability that of 6 vessels expected (1) exactly 5 will arrive safely, (2) at least 5 will arrive safely.

6. In tossing seven coins, what is the probability for each of the following number of heads: (*a*) seven; (*b*) six; (*c*) five; (*d*) four; (*e*) three; (*f*) two; (*g*) one; (*h*) zero.

7. In teams of two students, throw seven coins and make a mark in a scheme such as the following to score the number of heads:

7 heads........	3 heads........
6 heads........	2 heads........
5 heads........	1 head ....
4 heads........	0 heads........

Continue the experiment until you have scored 128 tossings of the seven coins. With what frequencies did you get 7 heads, 6 heads, 5 heads, $\cdots$ 0 heads out of a total of the 128 tossings?

8. In 128 trials each of which consists in tossing 7 coins, what is the *expected number* of occurrences (Arts. 164 and 166) of (*a*) 7 heads; (*b*) 6 heads; (*c*) 5 heads; (*d*) 4 heads; (*e*) 3 heads; (*f*) 2 heads; (*g*) 1 head; (*h*) 0 heads?

9. Discuss the deviations of the experimental results in problem 7 from the corresponding theoretical results in problem 8.

10. Five coins are tossed up. What is the probability of an odd number of heads?

11. A bag contains 5 white and 3 red balls. If 4 balls are drawn out one at a time and not replaced, find the probability that they are alternately of different colors.

12. The probability that A will win a certain game whenever he plays is $\frac{1}{3}$. If he plays 4 times, find the probability that he will win, (*a*) exactly twice, (*b*) at least twice, (*c*) at most twice.

13. Find the expectation of a man who buys a lottery ticket in a lottery of 100 tickets where there are four prizes of \$400, ten of \$200, and twenty of \$20.

14. If q be the probability of failure in a single trial, show that $1 - q^n$ is the probability of at least one success in n trials.

15. An Italian nobleman, interested in gambling, had, by continued observation of a game with three dice, noticed that the sum 10 appeared more often than the sum 9. He expressed his surprise at this to Galileo and asked for an explanation. Find the probability of (*a*) the sum 10, (*b*) the sum 9, and explain the difficulty of the nobleman.

16. According to the "United States Life Tables," 1910, out of 76,675 males living at the age of 25 years, 21,213 will be living at the age of 75. Out of 79,481 females living at age 25 years, 26,155 will be living at age 75. A husband and a wife are 25 each at the date of marriage; what is the probability that at least one of them will be living 50 years after the marriage?

17. Which is the greater, the probability of throwing at least one ace in six trials of throwing a die, or the probability of throwing at least one head of a coin in two trials?

18. A machinist works 300 days in a year. If the probability of his meeting with an accident on any particular work day is $\frac{1}{1000}$, show that the probability of his entirely escaping injury for a year is approximately $\frac{3}{4}$.

19. Find the probability of throwing six with a single die at least once in five trials.

CHAPTER XIX

PARTIAL FRACTIONS

167. Introduction. Early in the study of algebra we added together algebraic fractions and found the sum to be a single fraction whose denominator is the lowest common multiple of the denominators. Thus,

$$\frac{6}{x+1} + \frac{2}{2x+3} = \frac{14x+20}{2x^2+5x+3}.$$

It is often necessary to perform the inverse operation, that is, to decompose a given fraction into a sum of other fractions (called "partial" fractions) having denominators of lower degree. Thus, it is easily shown that $\frac{2x}{x^2-1}$ can be decomposed into $\frac{1}{x+1} + \frac{1}{x-1}$.

An algebraic fraction is said to be **proper** when its numerator is of lower degree than its denominator. In this chapter it is necessary to consider only proper fractions; for if the degree of the numerator is not lower than that of the denominator, the fraction may be reduced by division to the sum of an integral part and a proper fraction. Thus,

$$\frac{3x^4 - 3x^2 + 2x}{x^2 - 1} = 3x^2 + \frac{2x}{x^2-1}.$$

We shall *assume* the possibility of decomposing any proper fraction whose denominator contains factors prime to each other into the partial fractions of the types

$$\frac{A}{ax+b}, \quad \frac{B}{(ax+b)^p}, \quad \frac{Cx+D}{ax^2+bx+c}, \quad \frac{Ex+F}{(ax^2+bx+c)^q}$$

where $A, B, C, D, E, F, a, b, c$ are real numbers, p, q positive integers and ax^2+bx+c an expression without real linear factors.* With this assumption we shall show how to decompose certain classes of fractions.

168. Case I. *When the denominator can be resolved into factors of the first degree, all of which are real and different.*

* See Chrystal's *Algebra*, Fifth edition, Part I, Chapter VIII.

Example. Resolve $\dfrac{1 - x + 6x^2}{x - x^3}$ into its simplest partial fractions.

The sum of three fractions

$$\frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x}$$

will give a fraction whose denominator is $x - x^3$. We, therefore, try to determine A, B, and C so that

$$\frac{1 - x + 6x^2}{x - x^3} \equiv \frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x}$$
$$\equiv \frac{A(1 - x)(1 + x) + Bx(1 + x) + Cx(1 - x)}{x(1 + x)(1 - x)}.$$

Then,

$$1 - x + 6x^2 \equiv A(1 - x)(1 + x) + Bx(1 + x) + Cx(1 - x). \quad (1)$$

The two members of (1) are equal for all values of x except possibly for $x = 0$, $x = 1$, $x = -1$. Hence, by Art. 112, Corollary II, they are equal for these values. In (1), making

$x = 0$, we obtain $A = 1$;
making $x = 1$, we obtain $B = 3$;
making $x = -1$, we obtain $C = -4$.

Therefore, $\dfrac{1 - x + 6x^2}{x - x^3} \equiv \dfrac{1}{x} + \dfrac{3}{1 - x} - \dfrac{4}{1 + x}$.

Exercise. Verify by adding terms of second member.

The values of A, B, and C could also have been obtained by arranging the right-hand member of (1) in powers of x and equating coefficients of like powers (Art. 112, Corollary I); thus,

$$1 - x + 6x^2 = A + (B + C)x + (-A + B - C)x^2.$$

$$A = 1,$$
$$B + C = -1,$$
$$-A + B - C = 6.$$

These equations when solved yield $A = 1$, $B = 3$, $C = -4$.

In resolving a fraction into partial fractions, for every factor $(ax + b)$ occurring in the denominator there is a single partial fraction of the form $\dfrac{A}{(ax + b)}$ where A is a constant.

EXERCISES

Resolve each of the following into its simplest partial fractions and verify your result when the answer is not given:

1. $\dfrac{1}{x(x+1)}$.
2. $\dfrac{5x+1}{x^2+x-2}$.
3. $\dfrac{x-3}{6x^2-x-1}$.
4. $\dfrac{x}{2x^2-9x+9}$.
5. $\dfrac{10x+24}{5-24x-5x^2}$.
6. $\dfrac{14-47x}{4x-14x^2}$.
7. $\dfrac{x^3-4x-4}{x^2-4}$.
8. $\dfrac{12x^2+12x+1}{4x^2-1}$.
9. $\dfrac{x^2+13x+6}{(x+1)(x+2)(x-2)}$.
10. $\dfrac{6x^3-7x^2-13x-20}{3x^2-8x-3}$.
11. $\dfrac{6x^2+22x+18}{x^3+6x^2+11x+6}$.
12. $\dfrac{18x^2+10x+1}{6x^3+5x^2+x}$.

169. Case II. *When the denominator can be resolved into real linear factors, some of which are repeated.*

Example: Resolve $\dfrac{6x^3-8x^2-4x+1}{x^2(x-1)^2}$ into its simplest partial fractions.

The sum of four fractions

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$$

will give a fraction whose denominator is $x^2(x-1)^2$; we therefore try to determine A, B, C, D so that

$$\frac{6x^3-8x^2-4x+1}{x^2(x-1)^2}\equiv\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}.$$

Then,

$$\begin{aligned}6x^3-8x^2-4x+1 &\equiv Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2\\ &\equiv (A+C)x^3+(-2A+B-C+D)x^2\\ &\qquad +(A-2B)x+B.\end{aligned}$$

Equating coefficients of like powers (Art. 112, Corollary I) we have,

$$\begin{aligned}A+C&=6,\\ -2A+B-C+D&=-8,\\ A-2B&=-4,\\ B&=1.\end{aligned}$$

Solving these equations for A, B, C, D, we find

$$A = -2, \quad B = 1, \quad C = 8, \quad D = -5.$$

Hence,

$$\frac{6x^3 - 8x^2 - 4x + 1}{x^2(x-1)^2} \equiv \frac{-2}{x} + \frac{1}{x^2} + \frac{8}{x-1} - \frac{5}{(x-1)^2}.$$

In this case, for every factor $(ax + b)$ which occurs r times there are r partial fractions of the form

$$\frac{A_1}{ax+b}, \quad \frac{A_2}{(ax+b)^2}, \cdots, \quad \frac{A_r}{(ax+b)^r},$$

where A_1, A_2, $\cdots$, A_r are constants.

EXERCISES

Resolve each of the following into its simplest partial fractions and verify the result.

1. $\dfrac{2x+5}{(x+3)(x+1)^2}$.
2. $\dfrac{x^2+1}{2x(x-1)^2}$.
3. $\dfrac{5x^2 - x - 1}{x^2(x-1)}$.
4. $\dfrac{2x - 8}{x^2(3x-4)}$.
5. $\dfrac{12x^2 - 27x + 16}{x(3x-4)^2}$.
6. $\dfrac{x^3 - 3x^2 + 10x - 2}{(x+1)(x-1)^2}$.
7. $\dfrac{2x^4 + 3x^3 - 7x^2 + x + 4}{x^2(x+1)}$.
8. $\dfrac{2x^2+2}{(x+1)^2(x-1)^2}$.
9. $\dfrac{2x^3 + 2x^2 - 2x + 2}{(x+1)^2(x-1)^2}$.
10. $\dfrac{10x^3 - 29x^2 + 19x - 4}{x^2(1-2x)^2}$.

170. Case III. *When the denominator contains quadratic factors which are not repeated and which cannot be separated into real linear factors.*

Example: Resolve $\dfrac{11x^2 + 11x - 2}{(2x^2+x+1)(3x-2)}$ into a sum of partial fractions.

Let
$$\frac{11x^2 + 11x - 2}{(2x^2+x+1)(3x-2)} \equiv \frac{Ax+B}{2x^2+x+1} + \frac{C}{3x-2}.$$

Then
$$11x^2 + 11x - 2 \equiv (Ax+B)(3x-2) + C(2x^2+x+1)$$
$$\equiv (3A+2C)x^2 + (-2A+3B+C)x - 2B + C$$

Equating coefficients of like powers of x, we have

$$3A + 2C = 11,$$
$$-2A + 3B + C = 11,$$
$$-2B + C = -2,$$

whence $\quad A = 1, \quad B = 3, \quad C = 4,$

and $\dfrac{11x^2 + 11x - 2}{(2x^2 + x + 1)(3x - 2)} = \dfrac{x + 3}{2x^2 + x + 1} + \dfrac{4}{3x - 2}.$

In this case, for every factor $ax^2 + bx + c$ occurring once, there is a single partial fraction of the form $\dfrac{Ax + B}{ax^2 + bx + c}$, where A and B are real numbers.

EXERCISES

Resolve each of the following into its simplest partial fractions and verify the result.

1. $\dfrac{3x^2 - 2}{(x^2 + x + 1)(x + 1)}.$
2. $\dfrac{x + 1}{(x - 1)(x^2 + 1)}.$
3. $\dfrac{5}{x^3 + 5x}.$
4. $\dfrac{3}{x^3 - 1}.$
5. $\dfrac{3}{x^3 + 1}.$
6. $\dfrac{x^2 + 4x + 10}{x^3 + 2x^2 + 5}.$
7. $\dfrac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)}.$
8. $\dfrac{2x^4 + x^2 + 2x - 1}{x^3 + x}.$
9. $\dfrac{x^5 + x^4 + 5x^3 + 4x^2 + 3x + 2}{(x^2 + 1)(x^2 + x + 1)}.$
10. $\dfrac{x^2 + 2x + 9}{1 - x^4}.$

171. Case IV. *When the denominator contains quadratic factors which are repeated.*

Example: Resolve $\dfrac{16x^4 - 20x^3 + 14x^2 - 6x + 2}{(1 - 2x)(2x^2 - x + 1)^2}$ into partial fractions:

Solution: Let

$$\frac{16x^4 - 20x^3 + 14x^2 - 6x + 2}{(1 - 2x)(2x^2 - x + 1)^2} = \frac{A}{1 - 2x} + \frac{Bx + C}{2x^2 - x + 1} + \frac{Dx + E}{(2x^2 - x + 1)^2}.$$

Then $\quad 16x^4 - 20x^3 + 14x^2 - 6x + 2$

$$\equiv A(2x^2 - x + 1)^2 + (Bx + C)(1 - 2x)(2x^2 - x + 1) + (Dx + E)(1 - 2x)$$
$$\equiv (4A - 4B)x^4 + (-4A + 4B - 4C)x^3 + (5A - 3B + 4C - 2D)x^2 + (-2A + B - 3C + D - E)x + (C + E).$$

Equating coefficients of like powers of x, we have

$$\begin{aligned} 4A - 4B &= 16 \\ -4A + 4B - 4C &= -20 \\ 5A - 3B + 4C - 2D &= 14 \\ -2A + B - 3C + D - 2E &= -6 \\ A + C + E &= 2. \end{aligned}$$

Solving these equations for A, B, C, D, E, we find $A = +1$, $B = -3$, $C = 1$, $D = 2$, $E = 0$.

Hence,

$$\frac{16x^4-20x^3+14x^2-6x+2}{(1-2x)(2x^2-x+1)^2} = \frac{1}{1-2x} - \frac{3x-1}{2x^2-x+1} + \frac{2x}{(2x^2-x+1)^2}.$$

In this case, for every factor $(ax^2 + bx + c)$ occurring r times there are r partial fractions of the form

$$\frac{A_1x + B_1}{(ax^2 + bx + c)}, \quad \frac{A_2x + B_2}{(ax^2 + bx + c)^2} \cdots \frac{A_rx + B_r}{(ax^2 + bx + c)^r},$$

where $A_1, A_2, \cdots, A_r, B_1, B_2, \cdots, B_r$ are real numbers.

From the corollary of Art. 115, page 152 we know that any polynomial $f(x)$ with real coefficients can be expressed as a product of real linear and quadratic factors. Hence, if the factors of the denominator are known, any quotient of two polynomials with real coefficients can be decomposed into its partial fractions by the methods of this chapter.

EXERCISES

Resolve each of the following into its simplest partial fractions and verify your result when the answer is not given:

1. $\dfrac{2x^4 + 4x^2 + x + 2}{x(x^2 + 1)^2}$

2. $\dfrac{x^4 + 2x^3 + 2x^2 + 5x + 1}{x(x^2 + 1)^2}.$

3. $\dfrac{x^4 + x^3 - 2x^2 - 5x - 4}{(x - 1)(x^2 + x + 1)^2}.$

4. $\dfrac{x^4 + 18x^2 - 28x}{(x + 2)(x^2 - 2x + 4)^2}.$

5. $\dfrac{x^3 + 2x^2 + 2}{(x^2 + 1)^2}.$

6. $\dfrac{6x^4 + 15x^3 + 29x^2 + 23x + 16}{x(2x^2 + 3x + 4)^2}.$

7. $\dfrac{3x^5 - x^4 + 4x^3 + x^2 + 1}{(1 + x^2)^2(1 - x + x^2)}.$

8. $\dfrac{x^6 - x^5 + 5x^4 - 4x^3 + 3x^2 + 5x + 16}{x(1 - x)^2(x^2 + 4)^2}.$

9. $\dfrac{3x^4 + 3x^2 - x + 1}{x^3(x^2 + 1)^2}.$

10. $\dfrac{-3x^4 + 2x^2 + 1}{x^3(x^2 + 1)^2}.$

CHAPTER XX

DETERMINANTS

172. Extension of the determinant notation. Determinants of the second and third orders were used in Chapter V in the solution of systems of linear equations in two and three unknowns; and a determinant of the second order was so defined that the pair of values

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{1}$$

satisfies the system of equations,

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

provided

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \tag{2}$$

Analogously, a determinant of the third order was so defined that the set of values

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \tag{3}$$

satisfies the system of equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

provided

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0. \tag{4}$$

The determinant notation is extended in the present chapter to the solution of systems of linear equations containing more than three unknowns, and to certain problems of elimination.

It will be observed that each term in the expansions,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \tag{5}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2, \tag{6}$$

of determinants of orders 2 and 3 respectively, consists (except for sign) of the product formed by taking one and only one element from each row and column. This fact suggests the extension of determinants to represent certain expressions in n^2 elements by means of an array,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \cdots l_1 \\ a_2 & b_2 & c_2 & d_2 \cdots l_2 \\ a_3 & b_3 & c_3 & d_3 \cdots l_3 \\ a_4 & b_4 & c_4 & d_4 \cdots l_4 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ a_n & b_n & c_n & d_n \cdots l_n \end{vmatrix} \tag{7}$$

where the expansion is to consist of terms which are products formed by taking one and only one element from each row and column, and where the signs of terms are to be consistent with the special cases of $n = 2$ and $n = 3$ (Art. 172).

A square array such as (7) is called a determinant of the nth order. The diagonal from the upper left-hand to the lower right-hand corner of the square array is called the **principal diagonal** of the determinant, and the product, $a_1b_2c_3 \cdots l_n$, of the n numbers in this diagonal is called the **principal term** of the determinant.

173. Meaning of a determinant. In order to give the meaning of a determinant, we introduce the notion of an inversion. If, in an arrangement of positive integers, a greater precedes a less, there is said to be an **inversion.** Thus, in the order 12543, there are three inversions: 5 before 4, 5 before 3, 4 before 3. In 2341576, there are four inversions. When applied to any term in the expansion of a determinant such as (7), we say there is an inversion if the order of the subscripts presents an inversion when the letters (apart from subscripts) have the order $abcd \cdots l$ of the principal diagonal. With respect to determinants of orders 2 and 3, it may be observed that the number of inversions is even when the term is positive, and that the number of inversions is odd when the term is negative.

Consistent with these conditions, we lay down the following

DEFINITION. *A square array of n^2 elements is called a* **determinant of the nth order.** *It is an abbreviation for the algebraic sum of all the products that can be formed*

(1) *by taking as factors one and only one element from each column and each row of the array, and*

(2) *by giving to each term a positive or a negative sign according as the number of inversions of the subscripts of the term is even or odd, when the letters have the same order as in the principal diagonal.*

It may be added that if in any case the number of inversions in the principal diagonal is different from zero, the sign of a term is + or − according as the number of inversions in its subscripts differs from the number in the principal diagonal by an even or odd number. Since the subscripts fix the signs of terms, it may appear necessary to carry subscripts along in any numerical case, but we shall derive other modes of expansion (Arts. 174, 175) which make this unnecessary. We shall, in general, use the Greek letter Δ to represent a determinant.

ORAL EXERCISES

How many inversions are there in each of the following arrangements?

1. 1243.
2. 1423.
3. 1432.
4. 41352.
5. 21457368.
6. 21657348.

THEOREM. *The expansion of a determinant Δ of order n contains $n!$ terms.*

Since the number of terms is the same as the number of permutations of the subscripts $1, 2, 3, \cdots, n$, the number is $n!$ (Art. 153).

174. Useful properties of determinants. The following theorems embody useful properties of determinants.

THEOREM I. *If in a determinant Δ corresponding rows and columns are interchanged, the expansion is unchanged.*

Thus,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM II. *If two rows (or columns) of a determinant* Δ *are interchanged, the sign of the determinant is changed.*

Let us take for simplicity a determinant of the third order, but the argument used will clearly apply to any determinant. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}.$$

In the first place, interchanging two adjacent rows will simply interchange two adjacent subscripts in each term of the expansion. This will change the sign of every term of the expansion. Consider next the effect of interchanging any two rows (or columns) separated by m intermediate rows. The lower row can be brought just below the upper one by m interchanges of adjacent rows. To bring likewise the upper row into the original position of the lower row, $m + 1$ further interchanges are necessary. Hence, interchanging the two rows in question is equivalent to $2m + 1$ interchanges of adjacent rows. Since $2m + 1$ is an odd number, this process changes the sign of the determinant.

THEOREM III. *If a determinant* Δ *has two rows (or columns) identical, its value is zero.*

If we interchange two rows, we obtain, by Theorem II, $-\Delta$. But since the interchange of two identical rows does not alter the determinant we have
$$\Delta = -\Delta,$$
that is,
$$2\,\Delta = 0,$$
or
$$\Delta = 0.$$

THEOREM IV. *If all the elements of a row (or column) of* Δ *are multiplied by the same number m the determinant is multiplied by m.*

For, one element from the column multiplied by m must enter into each term of the expansion of Δ.

THEOREM V. *If one row (or column) of* Δ *has as elements the sum of two or more numbers,* Δ *can be written as the sum of two or more determinants.* That is,

$$\Delta = \begin{vmatrix} a_1 + a_1' + a_1'' & b_1 & c_1 \\ a_2 + a_2' + a_2'' & b_2 & c_2 \\ a_3 + a_3' + a_3'' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.$$

This theorem is evident for this special case, since each term in the expansion of Δ is evidently equal to the sum of the corresponding terms of the three determinants. Similarly, we can prove the general case.

THEOREM VI. *The value of any determinant Δ is not changed if each element of any row (or column), or each element multiplied by any given number, m, be added to the corresponding element of any other row (or column).*

By Theorems IV and VI,

$$\begin{vmatrix} a_1 + ma_3 & a_2 & a_3 \\ b_1 + mb_3 & b_2 & b_3 \\ c_1 + mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + m^* \begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix},$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0, \text{ by Theorem III.}$$

Likewise, the theorem can be proved for a determinant of any order.

The theorems of this article can often be used to good advantage in the simplification and evaluation of determinants.

Illustration 1.

Evaluate $\begin{vmatrix} 20 & 17 & 2 \\ 15 & 12 & 8 \\ 25 & 22 & -6 \end{vmatrix}.$

Solution: Factor out 5 from the first column, and 2 from the third and we have $5 \cdot 2 \begin{vmatrix} 4 & 17 & 1 \\ 3 & 12 & 4 \\ 5 & 22 & -3 \end{vmatrix}$ (Theo. IV).

Next, subtract 4 times the first column from the second and we have $5 \cdot 2 \begin{vmatrix} 4 & 1 & 1 \\ 3 & 0 & 4 \\ 5 & 2 & -3 \end{vmatrix}$ (Theo. VI).

Subtract the second column from the first and factor 3 out of the resulting first column and we have $5 \cdot 2 \cdot 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & -3 \end{vmatrix} = 30.$

Illustration 2.

Evaluate $\begin{vmatrix} a & b+c & 1 \\ b & a+c & 1 \\ c & a+b & 1 \end{vmatrix}.$

* This notation means that the determinant is multiplied by m.

Solution: Add the first column to the second and we have
$$\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} \quad \text{(Theo. VI).}$$

Factor out $(a + b + c)$ from the second column and we have
$$(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0 \quad \text{(Theo. III).}$$

EXERCISES

Find the value of each determinant.

1. $\begin{vmatrix} 6 & 4 & 4 \\ 6 & 3 & -1 \\ 3 & 7 & -1 \end{vmatrix}$. **2.** $\begin{vmatrix} -1 & 6 & 5 \\ -2 & 3 & 3 \\ -10 & 8 & 10 \end{vmatrix}$. **3.** $\begin{vmatrix} 2 & 2 & -1 \\ -1 & 5 & 4 \\ -3 & 3 & 5 \end{vmatrix}$.

175. Expansion by minors. If we suppress both the row and column to which any element, say c_k, of the determinant belongs, the unsuppressed elements form a determinant called the *first minor* of c_k, and which we shall denote by the capital letter C_k. Thus, in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minor of b_2 is
$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}.$$

A determinant Δ may be expressed in terms of the elements $c_1, c_2, \cdots, c_n$ of a column (or row) and their first minors as follows:

1. *Form the product of each element such as c_k in the column by the corresponding minor C_k.*

2. *Give each of the products thus formed a positive or a negative sign according as the sum of the number of the row and the number of the column containing c_k is even or odd.*

3. *Take the algebraic sum of these results. This sum is equal to Δ.*

Thus,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

If we can establish this theorem, we have a systematic method for expanding any determinant, since the first minors of Δ are again determinants which can be expressed in terms of their own minors. This process can be continued until we have the expansion of Δ.

The proof of the theorem involves two steps:

(1) The coefficient of the leading element a_1 in the expansion of Δ is the minor, A_1, of a_1. For, A_1 is a determinant of order $n-1$ in elements $b_2, b_3, \cdots, b_n, \cdots$ and its expansion therefore contains a term for each permutation of $2\ 3\ 4 \cdots n$. As to the signs of terms, the number of inversions is not changed by prefixing a_1.

(2) The coefficient of any element c_k in the expansion of Δ is its minor C_k with a $+$ or a $-$ sign, according as the sum of the number of the row and the number of the column containing c_k is even or odd. If c_k is in the hth column and kth row, we can bring it to the leading position (column 1, row 1) without disturbing the relative positions of elements not found in column h or row k. This is done by interchanging the column in which c_k stands with each preceding column in turn until c_k is in column 1, and the row in which c_k stands with each preceding row in turn until c_k is in row 1. In making these changes, the sign of the determinant is changed, $h-1+k-1=h+k-2$ times (Art. 174, Theo. II). Hence, if Δ' denotes this determinant with c_k as the leading letter,

$$\Delta' = (-1)^{h+k-2}\Delta = (-1)^{h+k}\Delta.$$

Let C'_k be the minor of c_k in Δ'. By (1), the sum of the terms in the expansion of Δ' which contain c_k is $c_kC'_k$. Since the minor of c_k in Δ' is the same as in Δ, the coefficient of c_k in the expansion of Δ is $(-1)^{h+k}C_k$. This establishes the second step.

EXERCISES

Expand and find the value of each of the following determinants.

1.
$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & -4 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{vmatrix}.$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & -4 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 & -4 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -4 \\ -1 & 1 & 1 \end{vmatrix}$$
$$- 1\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -4 \\ 2 & 1 & 0 \end{vmatrix} = 48.$$

2.
$$\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 4 \\ 7 & 8 & 9 & 7 \\ 1 & 2 & 3 & 1 \end{vmatrix}.$$

Hint: Subtracting column 4 from column 1 we have by Theorem VI, Art. 174.

$$\Delta = \begin{vmatrix} -3 & 2 & 3 & 4 \\ 0 & 5 & 6 & 4 \\ 0 & 8 & 9 & 7 \\ 0 & 2 & 3 & 1 \end{vmatrix} = -3\begin{vmatrix} 5 & 6 & 4 \\ 8 & 9 & 7 \\ 2 & 3 & 1 \end{vmatrix}.$$

3. $\begin{vmatrix} 1 & 2 & -4 \\ 2 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix}.$ **4.** $\begin{vmatrix} -2 & 4 & -1 \\ 1 & 0 & 2 \\ -3 & 5 & -4 \end{vmatrix}.$ **5.** $\begin{vmatrix} -1 & 6 & 5 \\ -2 & 3 & 3 \\ -6 & 8 & 10 \end{vmatrix}.$

6. $\begin{vmatrix} -4 & 3 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ -3 & 5 & 4 & -5 \end{vmatrix}.$ **7.** $\begin{vmatrix} 2 & -1 & 5 & 1 \\ -3 & 0 & -1 & 2 \\ -1 & 2 & -2 & 3 \\ 1 & 1 & 0 & -3 \end{vmatrix}.$

8. $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & -1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \end{vmatrix}.$ **9.** $\begin{vmatrix} 2 & 7 & 6 & 5 \\ 1 & 1 & 7 & 3 \\ 1 & 5 & 3 & 4 \\ 4 & 7 & 5 & 6 \end{vmatrix}.$

10. Show that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a).$

Hint: When $a = b$, two columns are identical so that Δ vanishes, and by the factor theorem, Art. 106, $a - b$ is a factor of Δ.

Factor each of the following determinants:

11. $\begin{vmatrix} 1 & a & b \\ 1 & a^2 & b^2 \\ 1 & a^3 & b^3 \end{vmatrix}.$ **12.** $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}.$ **13.** $\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}.$

176. We shall now establish a theorem of determinants which is useful in performing the eliminations required in the solution of equations in two or more unknowns.

THEOREM. *In expanding a determinant by minors with respect to a certain column (or row), if the elements of this column (or row) are replaced by the corresponding elements of some other column (or row), the resulting expression vanishes.*

For example, we have, by Art. 175,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4.$$

We are to prove that

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 = 0. \qquad (1)$$

The left member of (1) is equal to the expression of the determinant derived from Δ by replacing the column of a's by the b's with corresponding subscripts. But this gives a determinant with two columns identical, which therefore vanishes (Art. 174, Theorem III). The same method of proof can manifestly be applied to a determinant of any order.

177. Systems of linear equations containing the same number of equations as unknowns. In Chapter V, we used determinants to express the solution of simultaneous equations containing two and three unknowns. We are now in a position to make use of determinants to solve a system of n linear equations in n unknowns.

For simplicity of notation, take $n = 4$, and consider the system of equations

$$a_1x + b_1y + c_1z + d_1w = k_1, \qquad (1)$$
$$a_2x + b_2y + c_2z + d_2w = k_2, \qquad (2)$$
$$a_3x + b_3y + c_3z + d_3w = k_3, \qquad (3)$$
$$a_4x + b_4y + c_4z + d_4w = k_4, \qquad (4)$$

to be solved for x, y, z, and w if a solution exists. It is convenient to call the determinant of the coefficients of the unknowns,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

the *determinant of the system* of equations.

CASE I. When $\Delta \neq 0$.

As above, let $A_1, A_2, \cdots, B_1, B_2, \cdots$ be the minors of $a_1, a_2, \cdots, b_1, b_2, \cdots$ respectively. Multiplying both members of (1), (2), (3), and (4) by A_1, $-A_2$, A_3, and $-A_4$, respectively, we obtain

$$A_1a_1x + A_1b_1y + A_1c_1z + A_1d_1w = A_1k_1, \qquad (5)$$
$$-A_2a_2x - A_2b_2y - A_2c_2z - A_2d_2w = -A_2k_2, \qquad (6)$$
$$A_3a_3x + A_3b_3y + A_3c_3z + A_3d_3w = A_3k_3, \qquad (7)$$
$$-A_4a_4x - A_4b_4y - A_4c_4z - A_4d_4w = -A_4k_4. \qquad (8)$$

Adding (5), (6), (7), (8), we obtain Δ for the coefficient of x (Art. 175), and zero for coefficients of the other unknowns (Art. 176). That is,

$$\Delta \cdot x = A_1k_1 - A_2k_2 + A_3k_3 - A_4k_4. \qquad (9)$$

$$\text{Similarly,}\quad \Delta \cdot y = -B_1k_1 + B_2k_2 - B_3k_3 + B_4k_4, \tag{10}$$
$$\Delta \cdot z = C_1k_1 - C_2k_2 + C_3k_3 - C_4k_4, \tag{11}$$
$$\text{and}\quad \Delta \cdot w = -D_1k_1 + D_2k_2 - D_3k_3 + D_4k_4. \tag{12}$$

If, in Δ, we replace the a's by k's and expand, we have the right-hand member of (9). Similarly, replacing the b's, c's, and d's respectively by k's, we have the right-hand members of (10), (11), and (12). It follows that

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\Delta}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}}{\Delta},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}}{\Delta}, \quad w = \frac{\begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}}{\Delta}$$

is a solution, and the only solution, of (1), (2), (3), (4).

Hence, to obtain the solution of any system of n linear equations containing n unknowns when Δ, the determinant of the system, is not zero, we apply the

Theorem. *Any unknown is equal to a fraction (1) whose denominator is the determinant of the system, and (2) whose numerator is the determinant formed from the determinant of the system by substituting for the coefficients of the unknown sought the corresponding known terms with that sign attached to each known term which it has when on the side of the equation opposite the unknowns.*

Case II. When $\Delta = 0$.

If a solution exists when $\Delta = 0$, it cannot take the preceding form, since division by zero is excluded from algebraic operations. While the theory becomes too complicated in this case to be presented in full here, certain particular cases may well be considered.

As a rule (subject to certain exceptions), a system of equations has no solution when $\Delta = 0$. For example, the system

$$3x + 4y = 5,$$
$$6x + 8y = 9$$

has no solution. Likewise the system

$$x + y - z = 5,$$
$$4x + y - 2z = 9,$$
$$5x + 2y - 3z = 1$$

has no solution.

A system may, however, have an infinite number of solutions when $\Delta = 0$. For instance, the equations

$$x + y - z = 0, \tag{13}$$
$$4x + y - 2z = 0, \tag{14}$$
$$5x + 2y - 3z = 0 \tag{15}$$

constitute such a system. These equations are manifestly satisfied by $x = y = z = 0$. This is called the **trivial solution.** To obtain other solutions solve (13) and (14) for x and y in terms of z. This gives

$$x = \tfrac{1}{3}z, \qquad y = \tfrac{2}{3}z. \tag{16}$$

These values of x and y satisfy (15) as well as (13) and (14). Hence any value assigned to z with the corresponding values x and y obtained from $x = \frac{1}{3}z$, $y = \frac{2}{3}z$ satisfies (13), (14), and (15). Since z may have any value, there is an infinite number of solutions of the system in question.

Systems with an infinite number of solutions may be more generally illustrated by the **homogeneous * equations**

$$a_1x + b_1y + c_1z = 0, \tag{17}$$
$$a_2x + b_2y + c_2z = 0, \tag{18}$$
$$a_3x + b_3y + c_3z = 0, \tag{19}$$

when
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \tag{20}$$

but some minor of Δ is not zero, say $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$ (21)

To prove that (17), (18), (19) have an infinite number of solutions, substitute in (19) the values

$$x = \frac{\begin{vmatrix} -c_1z & b_1 \\ -c_2z & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & -c_1z \\ a_2 & -c_2z \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

* A homogeneous equation is one in which all terms are of the same degree in the unknowns.

which satisfy (17) and (18) when condition (21) is fulfilled. This substitution gives, after clearing of fractions,

$$-za_3\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} - zb_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + zc_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which, by (20), vanishes whatever value be assigned to z. Hence, z can take an infinite number of values, each of which with the corresponding x and y satisfies (17), (18), and (19).

178. Systems of equations containing more unknowns than equations. Consider first the single equation

$$3x + 5y - 6 = 0 \tag{1}$$

with two unknowns. It is clear from our work on graphs of equations (Art. 27) that there are an infinite number of pairs of values of x and y which satisfy this equation.

Consider next the two equations,

$$3x - 4y - 2z + 1 = 0, \tag{2}$$

$$4x + 3y - z - 6 = 0 \tag{3}$$

with three unknowns.

We may solve (2) and (3) for x and y in terms of z. This gives

$$x = \frac{\begin{vmatrix} 2z - 1 & -4 \\ z + 6 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -4 \\ 4 & 3 \end{vmatrix}} = \frac{10z + 21}{25}, \tag{4}$$

$$y = \frac{\begin{vmatrix} 3 & 2z - 1 \\ 4 & z + 6 \end{vmatrix}}{\begin{vmatrix} 3 & -4 \\ 4 & 3 \end{vmatrix}} = \frac{-5z + 22}{25}. \tag{5}$$

Any value assigned to z and the corresponding x and y obtained from (4) and (5) satisfy (2) and (3). Hence, the system has an infinite number of solutions.

The main point to be brought out by these illustrations is that, in general, from n equations containing more than n unknowns, we may solve (Art. 177) for some selected n of the unknowns in terms of the remaining unknowns. We are then at liberty to assign any values to these remaining unknowns, and thus obtain an infinite number of solutions. The problem in the exceptional cases in which it is impossible to solve for a selected set of n unknowns is too complicated to be treated here.

179. Systems of equations containing fewer unknowns than equations. Elimination. In a system of n linear equations taken at random, with m unknowns, $m < n$, the equations are usually inconsistent, that is, the solution of m of the equations will not satisfy the remaining equations. However, under certain conditions, all of the n equations are consistent.*

ORAL EXERCISES

1. Give three linear equations in x and y that are inconsistent.
2. Give three linear equations in x and y that are consistent.

We shall restrict our discussion of systems containing fewer unknowns than equations to the important case in which the number of equations is one greater than the number of unknowns.

Consider the equations

$$a_1x + b_1y + c_1 = 0, \tag{1}$$
$$a_2x + b_2y + c_2 = 0, \tag{2}$$
$$a_3x + b_3y + c_3 = 0. \tag{3}$$

If these three equations are consistent in case $a_1b_2 - a_2b_1 \neq 0$, then

$$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

which satisfy (1) and (2) must satisfy (3). This requires that

$$-a_3\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} - b_3\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + c_3 = 0.$$

Clearing of fractions, and interchanging columns in $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, we obtain

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

* Two or more equations are *consistent* (Art. 27) when they have a common solution.

$$\text{or,} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \quad \text{(Art. 175)} \tag{4}$$

as a condition to be satisfied in order that equations (1), (2), and (3) be consistent. The unknowns x and y are eliminated, and the determinant in (4) is called the **eliminant** of the equations (1), (2), and (3). Stated in words, in order that three linear equations in two unknowns have a common solution, it is necessary that the eliminant of the system shall be equal to zero.

The method used for three linear equations in two unknowns can be extended to any number n of linear equations in $n - 1$ unknowns. Thus, we have the

THEOREM. *The determinant (eliminant) formed of the coefficients of the unknowns and of the known terms must vanish in order that the n equations in $n - 1$ unknowns have a common solution.*

While the vanishing of the eliminant is a necessary condition for the existence of a common root, it is not a sufficient condition as is shown by the following example.

Take the system of equations

$$x + y - 4 = 0, \tag{5}$$

$$2x + 2y + 5 = 0, \tag{6}$$

$$x + y - 6 = 0. \tag{7}$$

$$\text{Here,} \quad \begin{vmatrix} 1 & 1 & -4 \\ 2 & 2 & 5 \\ 1 & 1 & -6 \end{vmatrix} = 0,$$

but any two of the equations are inconsistent.

In establishing the above necessary condition, we assumed that two of the equations have a solution. This condition is satisfied by no two of equations (5), (6), (7).

EXERCISES

Solve by using determinants.

1. $4x - 3y = 5,$
$8x + y = 17.$

2. $3x + 4y - 2z - 5 = 0,$
$4x - 3y + 8z + 4 = 0,$
$2x + 8y - 3z = 5.$

3. $\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = 1,$
$\dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = 1,$
$\dfrac{1}{4x} + \dfrac{1}{4y} + \dfrac{1}{2z} = 1.$

4. $$\begin{aligned} 3x + 2y + 4z - w &= 13, \\ 5x + y - z + 2w &= 9, \\ 2x + 3y - 7z + 3w &= 14, \\ 4x - 4y + 3z - 5w &= 4. \end{aligned}$$

5. $$\begin{aligned} x + y + z + w &= 0, \\ 3x - 4y + 5z + 6w &= 1, \\ x - 2y + 3z - 4w &= 29, \\ 2x + 3y - 4z - 5w &= 9. \end{aligned}$$

6. Find a value of k such that

$$\begin{aligned} kx - 3y - 5 &= 0, \\ 8x + y - 17 &= 0, \\ kx + 2y - 10 &= 0 \end{aligned}$$

are consistent equations. Can k take more than one value?

7. By Art. 179, it follows that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ is a necessary condition that the two equations

$$\begin{aligned} a_1x + b_1 &= 0, \ (a_1 \neq 0) \\ a_2x + b_2 &= 0 \ \ (a_2 \neq 0) \end{aligned}$$

be consistent. Show that this condition is also sufficient for this special case.

8. Discuss the number of values of x, y, z which satisfy

$$\begin{aligned} x + 3y - z &= 0, \\ -2y + z &= 0, \\ 5x + y + 2z &= 0, \end{aligned}$$

and find the ratios $x : y : z$ of corresponding values apart from the trivial solution $x = y = z = 0$.

Prove each system of equations inconsistent or find a solution of the system.

9. $$\begin{aligned} x + y &= 7, \\ x + 2y &= 10, \\ 3x + 4y &= 25. \end{aligned}$$

10. $$\begin{aligned} 3x + 4y &= 25, \\ x + y &= 7, \\ x + 2y &= 11. \end{aligned}$$

11. $$\begin{aligned} x - y &= 1, \\ 3x - y &= 5, \\ 2x + 3y &= 7. \end{aligned}$$

12. If (x, y, z) is a non-trivial solution of the system

$$\begin{aligned} 3x - 4y &= 7z, \\ 4x + y &= 3z, \end{aligned}$$

show that $$x : y : z = 1 : -1 : 1.$$

13. Eliminate x and y from the equations $y = m_1x + b_1$, $y = m_2x + b_2$, $y = m_3x + b_3$, and show that the eliminant is zero if $b_1 = b_2 = b_3$.

Write the eliminant of each system of equations. Apply the theorem of Art. 179 to test whether the system satisfies our necessary condition for a solution. In case there is a solution, find it.

14. $$\begin{aligned} 2x + 7y &= 1, \\ x + y &= 3, \\ 2x - 3y &= 11. \end{aligned}$$

15. $$\begin{aligned} x + y + z &= 4, \\ 2x + 3y - 4z &= 17, \\ 6x - 3y - 3 &= 0, \\ x + 2y + 3z &= 2. \end{aligned}$$

CHAPTER XXI

LIMITS

180. Absolute values. The numerical value of x, that is, the value of x without regard to sign, is often represented by the symbol $|x|$ which is read "absolute value of x." In dealing with absolute values in this chapter it is convenient to emphasize two properties of absolute values.

1. The absolute value of a sum is never greater than the sum of the absolute values of the numbers. (See Art. 68, exercise 13.)

For example, $|-7+3| \leqq |-7| + |3|,$

or $4 \leqq 10.$

2. The product of absolute values is equal to the absolute value of the product.

For example $|-7| \cdot |3| = |-7 \cdot 3|$

or $21 = 21.$

181. Definition of a limit. *A variable x is said to approach a constant a as a limit if $|a - x|$ becomes and remains less than any assigned positive number d when the variable x takes all values for which it is defined in the neighborhood of a.*

We have had many illustrations of limits in elementary mathematics. Thus, in geometry the area of a circle is considered as the limiting value of the area of the inscribed regular polygon as the number of sides is indefinitely increased. In this case, the values of the variable are the areas of the inscribed polygons as the number of sides is increased. Again, as we annex 3's to the decimal $.3333\cdots$, its value runs through the sequence of numbers .3, .33, .333, etc., which can be made to approach as near to $\frac{1}{3}$ as we please. In the geometrical progression

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots,$$

S_n, the sum of the first n terms, runs through the sequence

$$1, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{15}{8}, \cdots,$$

and approaches the limiting value 2.

The essence of the definition of a limit lies in the words "becomes and remains less." For example, if x runs through the sequence of values $\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \cdots,$
the difference $|1 - x|$ takes on the values

$$\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{1}{4}, \tfrac{5}{4}, \cdots$$

and becomes less than any assigned number but it does not remain so. In this case we cannot say that x approaches 1 as a limit.

To indicate that x approaches a as a limit, we use the notation

$$x \to a, \text{ or } \lim x = a.$$

182. Infinitesimals. A very important class of variables consists of those which have the limit zero. They are called **infinitesimals.** The area between a circle and the inscribed regular polygon as the number of sides increases, the weight of the air in the receiver of a perfectly working air pump, and the difference $2 - S_n$, where S_n is the sum of the first n terms of the series $1 + \frac{1}{2} + \frac{1}{4} \cdots$, are examples of infinitesimals.

THEOREM. *If $u \to 0$ and $v \to 0$, and $|X|$ and $|Y|$ are always less than some positive constant k, then $Xu + Yv \to 0$.*

In other words, *if u and v are infinitesimals, then $Xu + Yv$ is an infinitesimal.*

Let d be any positive number however small. Since $\lim u = 0$, and $\lim v = 0$, $|u|$ and $|v|$ will ultimately become and remain less than $\frac{d}{2k}$. For these values of u and v, we have

$$|X| \cdot |u| < |X| \frac{d}{2k}$$

and from (2) Art. 180

$$|Xu| < \frac{|X|d}{2k}.$$

In a similar way

$$|Yv| < \frac{|Y|d}{2k}.$$

Adding these two inequalities,

$$|Xu| + |Yv| < \frac{(|X| + |Y|)d}{2k}.$$

By hypothesis, $|X| + |Y| < 2k$,

hence, $|Xu| + |Yv| < d.$

But from (1), Art. 180

$$| Xu + Yv | \leqq | Xu | + | Yv |$$

or $$| Xu + Yv | < d.$$

Since d may be chosen as small as we please,

$$Xu + Yv \rightarrow 0.$$

This theorem may be extended to any number of variables.

COROLLARY. *If $u \rightarrow 0$ and $v \rightarrow 0$, and C is a constant, then*

$$Xu + Yv + C \rightarrow C.$$

Examples: If $u \rightarrow 0$ and $v \rightarrow 0$, then

(1) $$7u + 3v \rightarrow 0,$$

(2) $$8 - 3u + 5v \rightarrow 8,$$

(3) $$a + b - (u + v) \rightarrow a + b,$$

(4) $(a - u)(b - v) = ab - bu - av + uv = ab - (a - u)v - bu \rightarrow ab.$

183. Theorems concerning limits. The following theorems follow directly from the theorem of Art. 182.

THEOREM I. *The limit of the sum of two variables is the sum of their limits.*

Let the variables be x and y, and let

$$\lim x = a, \quad \lim y = b.$$

Then, $$x = a - u, \quad y = b - v,$$

where $$u \rightarrow 0, \quad v \rightarrow 0.$$

Adding, we have $\quad x + y = a + b - (u + v).$

From the corollary of Art. 182, $\;a + b - (u + v) \rightarrow a + b,$

or $$\lim (x + y) = a + b = \lim x + \lim y.$$

COROLLARY I. *The limit of the sum of any finite number of variables is the sum of their limits.*

COROLLARY II. *The limit of the difference of two variables is the difference of their limits.*

THEOREM II. *The limit of the product of two variables is the product of their limits.*

Using the notation of Theorem I,

$$xy = (a - u)(b - v) = ab - [(a - u)v + bu].$$

From the theorem of Art. 182,

$$(a - u)v + bu \to 0.$$

Hence, $$\lim xy = ab = \lim x \lim y.$$

COROLLARY I. *The limit of the product of any finite number of variables is the product of their limits.*

COROLLARY II. *If n is a positive integer,*

$$\lim x^n = a^n = (\lim x)^n.$$

COROLLARY III. *If c is any constant,*

$$\lim cx = c \lim x.$$

184. Both numerator and denominator with limit zero. If both the numerator and the denominator of a fraction $\frac{x}{y}$ approach the limit zero, we have a rather curious result, as is shown by the cases which occur in the following example.

In the fraction $\frac{x}{y}$ let y approach 0 through the sequence of values

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n}, \cdots.$$

Let x approach 0 through one of the four sequences:

(a) $$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \cdots, \frac{1}{4^n}, \cdots.$$

(b) $$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \cdots, \frac{1}{\sqrt{2^n}}, \cdots.$$

(c) $$\frac{k}{2}, \frac{k}{2^2}, \frac{k}{2^3}, \cdots, \frac{k}{2^n}, \cdots. \quad (k = \text{any constant.})$$

(d) $$\frac{1}{2}, -\frac{1}{2^2}, \frac{1}{2^3}, -\frac{1}{2^4}, \cdots, \pm\frac{1}{2^n}, \cdots.$$

CASE (a). We have here $\lim \frac{x}{y}$ = limit of $\dfrac{\frac{1}{4^n}}{\frac{1}{2^n}}$ as n increases without limit. Since $\dfrac{\frac{1}{4^n}}{\frac{1}{2^n}}$ reduces to $\frac{1}{2^n}$, $\lim \frac{x}{y}$ becomes $\lim \frac{1}{2^n} = 0.$

CASE (*b*). Here $\frac{x}{y}$ passes through the sequences of values

$$\sqrt{2},\ \sqrt{2^2},\ \sqrt{2^3},\ \cdots,\ \sqrt{2^n},\ \cdots$$

which increases without limit.

CASE (*c*). In this case $\lim \frac{x}{y} = k$.

CASE (*d*). Here $\frac{x}{y}$ takes alternately the values $+1$ or -1 and approaches no limit.

We see then that if x and y both approach 0 as a limit, their ratio may approach any number whatever including 0, may increase without limit, or may oscillate between two fixed numbers.

185. Infinity. If the numerator of the fraction is constant, or has the limit $a (a \neq 0)$, while the denominator has the limit 0, then $\frac{x}{y}$ increases without limit and is said to become **infinite.** This is usually expressed by writing $\lim_{y\to 0} \frac{x}{y} = \infty$.

It is not, however, to be inferred that infinity is a limit. The variable $\frac{x}{y}$ in the case just given does not approach a limit. If z is a variable which increases without limit, the various expressions "$\lim z = \infty$," "$z \to \infty$," "$z = \infty$," should not be read "z approaches infinity" or "z equals infinity," but "z becomes infinite," "z increases without limit." Infinity is not a number in the sense in which we are using the term.

THEOREM I. $\lim_{n\to\infty} \frac{1}{n} = 0.$

Let d be any assigned small positive number. Let $n \geqq \frac{x}{d}$ where x is any number greater than 1. Then $\frac{1}{n} \leqq \frac{d}{x} < d$. (IV, Art. 68.) That is, $\frac{1}{n}$ becomes and remains less than any assigned number.

THEOREM II. If $| r | < 1$, $\lim_{n\to\infty} r^n = 0$.

Since $| r | < 1$, it can be written in the form $| r | = \frac{1}{1+h}$ where h is positive. Hence,

$$| r^n | = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + \text{positive terms}}.$$

(By Binomial Theorem.)

Therefore $$| r^n | < \frac{1}{1 + nh} < \frac{1}{nh}.$$

By Theorem I of the present article and Corollary III, Art. 183,

$$\lim_{n\to\infty} \frac{1}{nh} = 0.$$

Hence, $$\lim_{n\to\infty} | r^n | = 0.$$

Since $r^n = \pm | r^n |$, we have

$$\lim_{n\to\infty} r^n = 0.$$

COROLLARY. *If* $| r | < 1$, $\lim_{n\to\infty} \frac{ar^n}{1-r} = 0.$

Exercise. Let y approach 0 through the sequence

$$0.1, 0.01, 0.001, \cdots.$$

Show that the fraction $\frac{x}{y}$ may be made to approach any number as a limit, may increase without limit, or may oscillate between two numbers.

186. Limiting value of a function. Let $f(x)$ represent any function of x. If x approaches a limit a and at the same time $f(x)$ takes on corresponding values such that

$$\lim f(x) = A,$$

we may abbreviate and write

$$\lim_{x\to a} f(x) = A,$$

which reads, "As x approaches a as a limit, $f(x)$ approaches the limit A"; or, more briefly, "The limit of $f(x)$, when x approaches a, is A."

If $$f(a) = \lim_{x\to a} f(x),$$

the function is said to be **continuous** for $x = a$.

187. Indeterminate forms. To find the value of the fraction $\frac{x^2 + x - 2}{x - 1}$ when $x = 2$, we substitute and find the value to be 4. But when $x = 1$, by substitution we find $\frac{0}{0}$, a meaningless symbol.

We may write $$\frac{x^2 + x - 2}{x - 1} = x + 2,$$

but since division by zero is excluded from our operations, this simplification does not hold for $x = 1$. But for every other value of x, however near to 1, the division is possible. Hence, letting x approach 1, we have

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} (x + 2) = 3.$$

Although substitution of $x = 1$ in $\frac{x^2 + x - 2}{x - 1}$ gives us a meaningless symbol, it is convenient to assign a value to the fraction. When $x = 1$, we *define* $\frac{x^2 + x - 2}{x - 1}$ to be $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3$. Giving this value to the fraction makes $\frac{x^2 + x - 2}{x - 1} = x + 2$ true for all values of x. In general, if $f(x)$ is a fraction which for $x = a$ takes the form $\frac{0}{0}$, we define $f(a)$ to be $\lim_{x \to a} f(x)$.

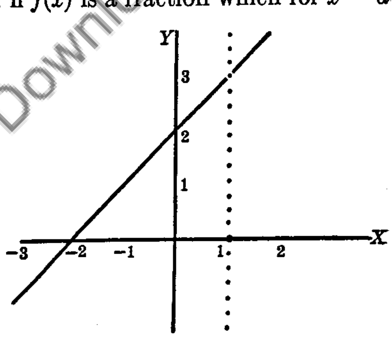

Fig. 45

The student should note that this is not a *necessary* definition of $f(a)$, but merely a convenient one. The convenience arises from the fact that with such a definition of $f(a)$, the function becomes continuous at $x = a$.

The above argument may be put in geometrical language. If we represent $y = \frac{x^2 + x - 2}{x - 1}$ graphically we have the result shown in Fig. 45, which is a straight line with a gap in it for $x = 1$. Since the function $\frac{x^2 + x - 2}{x - 1}$ is not defined for $x = 1$ we are at liberty to choose any one of the dots in the vertical line through $x = 1$ to represent $\dfrac{x^2 + x - 2}{x - 1}$ for $x = 1$. It is most convenient to choose the point which fills up the gap and makes the graph continuous. This is a geometrical version of the statement that it is convenient to define $f(a)$ as the $\lim\limits_{x \to a} f(x)$ when $f(x)$ takes on the form $\dfrac{0}{0}$ when $x = a$.

We wish sometimes to find the limit of the value of a function as the variable increases without limit. The following example illustrates the method.

Find the limit of $\dfrac{x^2 + 2}{3x^2 + 2x - 1}$ for $x \to \infty$.

By the theorems on limits this will take the meaningless form $\dfrac{\infty}{\infty}$, but dividing numerator and denominator by x^2, we can write the fraction as

$$\frac{1 + \dfrac{2}{x^2}}{3 + \dfrac{2}{x} - \dfrac{1}{x^2}},$$

and since $\dfrac{2}{x^2}, \dfrac{2}{x}, \dfrac{1}{x^2}$ are infinitesimals by Theorem I, Art. 185, we have $\frac{1}{3}$ for the limit of the fraction.

The symbols $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ are called indeterminate forms. Among other such forms which may arise are $0 \cdot \infty$ and $\infty - \infty$, but the expressions which give rise to these forms may be reduced to the form $\dfrac{0}{0}$, as shown in the following examples.

Example 1. $(x^2 + x - 2) \cdot \dfrac{1}{x - 1}$ takes the form $0 \cdot \infty$ when $x = 1$. For any other value of x we may write

$$(x^2 + x - 2) \cdot \frac{1}{x - 1} = \frac{x^2 + x - 2}{x - 1} = x + 2.$$

Hence, $\lim\limits_{x \to 1} (x^2 + x - 2) \cdot \dfrac{1}{x - 1} = \lim\limits_{x \to 1} (x + 2) = 3.$

Example 2. $\dfrac{x-1}{x^2-9} - \dfrac{x^2+x-6}{x(x^2-9)}$ takes the form $\infty - \infty$ when $x = 3$. For any other value of x, we may write

$$\frac{x-1}{x^2-9} - \frac{x^2+x-6}{x(x^2-9)} = \frac{-2(x-3)}{x(x^2-9)} = \frac{-2}{x(x+3)}.$$

Hence, $$\lim_{x\to 3}\left(\frac{x-1}{x^2-9} - \frac{x^2+x-6}{x(x^2-9)}\right) = \lim_{x\to 3}\frac{-2}{x(x+3)} = -\frac{1}{9}.$$

EXERCISES

What values should be given to the following expressions in order to make them continuous for the values of x indicated?

1. $\dfrac{x^2-9}{x-3}$ when $x = 3$.
2. $\dfrac{3x^2-4x+1}{x^2-1}$ when $x = 1$.
3. $\dfrac{x^3-27}{x-3}$ when $x = 3$.
4. $\dfrac{x^3-27}{x^2-9}$ when $x = 3$.
5. $\dfrac{x^3+a^3}{x^2-a^2}$ when $x = -a$.
6. $\dfrac{2x^2-5x-12}{x^2-5x+4}$ when $x = 4$.
7. $\dfrac{x^2-8x-9}{\sqrt{x}-3}$ when $x = 9$.
8. $\dfrac{2x^2+13x-7}{x^2+9x+14}$ when $x = -7$.
9. $(2x^2-3x)\cdot\left(\dfrac{2x+3}{x}\right)$ when $x = 0$.
10. $\dfrac{5-2x}{1-x^2} - \dfrac{3}{x(1-x^2)}$ when $x = 1$.

As n increases without limit find the limits of the following fractions.

11. $\dfrac{3+4n}{n}$.
12. $\dfrac{3n}{2n^2-1}$.
13. $\dfrac{5n^3-n}{3n^3+7n^2}$.
14. $\dfrac{2x^2+3x-7}{0.3x^2-x+9}$.

CHAPTER XXII

INFINITE SERIES

188. Definition. Let $u_1, u_2, \cdots, u_n, \cdots$ be any unending sequence of real numbers positive or negative. The expression

$$u_1 + u_2 + \cdots + u_n + u_{n+1} + \cdots,$$

when the terms are formed according to some law of succession, is called an **infinite series.**

In the discussion of geometric progressions (Art. 86) we have met such series, for example,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots.$$

189. Convergence and divergence. In the series

$$u_1 + u_2 + \cdots + u_n + \cdots,$$

let S_n represent the sum of the first n terms; that is, let

$$\begin{aligned} S_1 &= u_1, \\ S_2 &= u_1 + u_2, \\ S_3 &= u_1 + u_2 + u_3, \\ &\cdots\cdots \\ S_n &= u_1 + u_2 + \cdots + u_n. \end{aligned}$$

Example 1. In the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ given in Art. 188, we have

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1 + \frac{1}{2} = \frac{3}{2}, \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}, \\ &\cdots\cdots \\ S_n &= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}. \end{aligned}$$

Example 2. In the series, $1 + 2 + 3 + 4 + \cdots$, we have

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1 + 2 = 3, \\ S_3 &= 1 + 2 + 3 = 6, \\ &\cdots\cdots \\ S_n &= 1 + 2 + 3 + \cdots + n = \frac{n}{2}(1 + n). \end{aligned}$$

Example 3. In the series, $1 - 1 + 1 - 1 + 1 - \cdots$, we have

$$S_1 = 1,$$
$$S_2 = 0,$$
$$S_3 = 1,$$
$$\cdots\cdots\cdots$$
$$S_n = 1 \text{ or } 0, \text{ according as } n \text{ is odd or even.}$$

These three examples illustrate three cases which may occur.

CASE I. S_n approaches a limit, say S, as n increases without limit. In the first example above, S_n is never greater than 2, no matter how large a number n represents and approaches 2 as a limit, when n increases without limit.

CASE II. S_n is numerically larger than any assigned number for a sufficiently large value of n.

This case is illustrated in example 2.

CASE III. S_n remains finite but does not approach a limit as n increases without limit.

This case is illustrated in example 3, where S_n may have either of the values 0 or 1, according as n is even or odd.

Series which come under Case I are called **convergent** series and are by far the most important. Series which are included in Cases II and III are called **divergent** series. We have then the

DEFINITION. *When in an infinite series the sum of the first n terms approaches a limit as n increases without limit, the series is said to be convergent; otherwise it is divergent.*

The *limit, S, of the sum* of n terms of a convergent series, written $\lim_{n\to\infty} S_n$, is often called the *sum* * of the series. In connection with convergent series we shall also use the expression "limiting value of the series" to mean $\lim S_n$.

From the definition of convergence and certain theorems on limits (Art. 183), the following theorems may be stated.

THEOREM I. *A necessary condition for the convergence of an infinite series is that its nth term shall approach zero as a limit when* $n \to \infty$.

* The word "sum" is here used in a purely conventional sense. It is not to be understood as the sum of an infinite number of terms, but as the limit of the sum of n terms as n increases without limit.

For, $u_n = S_n - S_{n-1}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}, \text{ (Cor. II, Theo. I, Art. 183.)}$$
$$= S - S = 0.$$

That the necessary condition $\lim_{n\to\infty} u_n = 0$ is not a sufficient condition for convergence is well illustrated by the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

In this case, $\lim_{n\to\infty} \frac{1}{n} = 0$, but the series is divergent as will be shown in example 6, Art. 192.

Corollary. If $\lim_{n\to\infty} u_n$ is not zero, the series is divergent. For example, the series $\frac{1}{3} + \frac{2}{5} + \cdots + \frac{n}{2n+1} +$ is divergent, for the nth term has the limit $\frac{1}{2}$ as $n \to \infty$.

Theorem II. *The convergence of a series is not changed by the omission of a finite number of terms.*

For the sum of the terms omitted is a definite number which added to the sum of the new series gives a definite number for the sum of the entire series.

For an illustration, see examples 2 and 3, Art. 192.

EXERCISES

In exercises 1–3, the nth term of a series is given. Write the first three terms and the $(n + 1)$st term.

1. $\frac{1}{n}$. **2.** $\frac{n}{n+1}$. **3.** $\frac{(-1)^n x^n}{n!}$.

Write the nth term of each of the following series:

4. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$. **6.** $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$.

5. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$. **7.** $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

Show that each of the following series is divergent:

8. $\frac{1}{3} + \frac{2}{5} + \cdots + \frac{n}{2n+1} + \cdots$. **9.** $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \cdots$.

10. $a + ar + ar^2 + ar^3 + \cdots$ where $r > 1$.

SERIES OF POSITIVE TERMS

190. Fundamental assumption. *An infinite series of positive terms is convergent if S_n is always less than some definite number, however great n may be.**

Let K be a number such that $S_n < K$ for all values of n. Since the series contains positive terms only, S_n is a variable which increases as n increases. Since it can never attain so great a value as K, we assume that there is some number less than K which S_n approaches as a limit.

To illustrate this assumption, consider the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots,$$

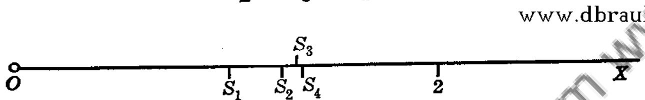

Fig. 46

and take points on the line OX (Fig. 46), to represent $S_1, S_2, S_3, \cdots$ so that the measure of OS_1, is S_1, of OS_2, is S_2, etc.

$$S_1 = 1,$$

$$S_2 = 1 + \frac{1}{2^2} = 1.2500,$$

$$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^3} = 1.2870,$$

$$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} = 1.2909,$$

. . . .

We can show that the sum of n terms of this series is less than 2. (See Art. 192, example 2.) Hence, according to the assumption of this section, there is some point not farther to the right than 2 which S_n approaches as a limit when $n \to \infty$.

An analogous assumption exists for a series all of whose terms are negative. An infinite series of negative terms is convergent if S_n is always algebraically greater than some definite number, however great n may be.

* For proof see Pierpont's *Theory of Functions of a Real Variable*, Art. 109.

EXERCISES

1. It can be shown that the sum of the series,

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{(n-1)!} + \cdots,^*$$

is always less than 3. Illustrate the assumption of this section graphically by means of this series.

2. Illustrate graphically the assumption for a series of negative terms by means of the series,

$$-1 - \frac{1}{2^2} - \frac{1}{3^3} - \frac{1}{4^4} - \cdots.$$

191. Comparison tests for convergence and divergence.

THEOREM I. *Given a series of positive terms*

$$u_1 + u_2 + \cdots + u_n + \cdots$$

to be tested for convergence. Suppose we find a series of positive terms

$$v_1 + v_2 + \cdots + v_n + \cdots$$

known to be convergent, and such that each term of the v-series is equal to or greater than the corresponding term of the u-series, then the u-series is convergent.

Let $$S_n = u_1 + u_2 + \cdots + u_n \qquad (1)$$

and $$S_n' = v_1 + v_2 + \cdots + v_n. \qquad (2)$$

By hypothesis,

$$u_1 \leqq v_1,\ u_2 \leqq v_2,\ \cdots,\ u_n \leqq v_n. \qquad (3)$$

Adding members of (3), and using (1) and (2), we have

$$S_n \leqq S_n'.$$

Then $$\lim_{n\to\infty} S_n \leqq \lim_{n\to\infty} S_n'.$$

Since by hypothesis lim S_n' exists, it follows that S_n is always less than a definite number, and by the assumption of Art. 190 the series $u_1 + u_2 + \cdots + u_n + \cdots$ is convergent.

THEOREM II. *Given a series of positive terms*

$$u_1 + u_2 + \cdots + u_n + \cdots$$

to be tested for divergence. Suppose we find a series of positive terms

$$v_1 + v_2 + \cdots + v_n + \cdots$$

* For meaning of 1!, 2!, 3!, etc. see Art. 92.

known to be divergent, and such that each term of the v-series is equal to or less than the corresponding term of the u-series, then the u-series is divergent.

Since $v_1 \leqq u_1$, $v_2 \leqq u_2$, $\cdots$, $v_n \leqq u_n$, if the u-series were convergent, the v-series would be convergent by Theorem I, but this is contrary to the hypothesis. Hence, the u-series is divergent.

192. Some examples of series useful in making comparison tests.

Example 1. The infinite geometric progression

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

is one of the most useful comparison series for testing convergence and divergence. This series converges if $|r| < 1$ and diverges if $|r| \geqq 1$. That it converges if $|r| < 1$ was proved in Arts. 86 and 185. By the corollary of Theorem I, Art. 189, the series diverges if $|r| \geqq 1$, since in this case ar^{n-1} does not approach zero as a limit as $n \to \infty$.

Example 2. Prove the series

$$2 + 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{(n-1)^{n-1}} + \cdots$$

to be convergent.

Solution: For purposes of comparison take as the v-series the geometric progression

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

Write the series to be tested under the comparison series, and we have

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots,$$

$$2 + 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{(n-1)^{n-1}} + \cdots.$$

After the third term, each term in the second series is less than the corresponding term just above it. That this is true for every term after the third is shown by examining the two nth terms. If $n > 3$, then

$$\frac{1}{(n-1)^{n-1}} < \frac{1}{2^{n-1}}.$$

Beginning with the fourth term, the sum of n terms of the first series is always less than $\frac{1}{4}$. Hence, the sum of the second series can never exceed $2 + 1 + \frac{1}{2^2} + \frac{1}{4} = 3\frac{1}{2}$. In comparing two series it is not sufficient to compare a few terms at the beginning of the series. The nth terms should be compared.

Example 3. Test for convergence the series

$$\frac{3 \cdot 3^3}{1} + \frac{2 \cdot 3^2}{1} + \frac{3}{1} + 1 + \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \cdots.$$

If we drop out the first three terms of this series and prove the remainder to be convergent, the given series must converge by Theorem II, Art. 189. Thus, if the series beginning with the fourth term has a sum, the sum of the entire series will be the sum of terms after the third plus $3 \cdot 3^3 + 2 \cdot 3^2 + 3 = 102$. Beginning then with the fourth term and comparing with the series,

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

which is known to converge to $\frac{3}{2}$ (Art. 86), we have

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n-1}} + \cdots,$$

and $$1 + \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \cdots + \frac{1}{(n-1) \cdot 3^{n-1}} + \cdots.$$

Each term in the second row is equal to or less than the corresponding term in the first. Hence, the second series converges to some number not greater than $\frac{3}{2}$ and the sum of the entire series in question is not greater than 103.5.

Example 4. Find the sum of the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots.$

Solution: Write S_n in the form

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Hence, $$\lim_{n \to \infty} S_n = 1,$$

and thus the series is convergent.

Example 5. Show that $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$, where $p > 1$, is convergent.

Solution: Write down the inequalities,

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}},$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{4^{p-1}},$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{8^{p-1}}.$$

. . . .

Add the members of the inequalities, thus

$$\frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots < \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots.$$

The right-hand side of this inequality is a geometric progression whose ratio is $\frac{1}{2^{p-1}}$, which is < 1 when $p > 1$. Hence, the series is convergent.

Example 6. Show that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution: Consider the inequalities:

$$1 + \frac{1}{2} > 1,$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2},$$

$$\cdots\cdots\cdots\cdots$$

Adding members of the inequalities, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

But the right-hand member of this inequality can be made as large as we please. The series in question is therefore divergent.

193. Summary of standard comparison series. When any new series has been shown to be convergent or divergent, we evidently increase our supply of series for comparison purposes, but the standard series given in examples 1–6, Art. 192 are so useful as to deserve special mention and are sometimes called **comparison series.**

Convergent series for comparison.

1. $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \ (r < 1)$. (exercise 1, Art. 192)

2. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$. (exercise 4, Art. 192)

3. $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \ (p > 1)$. (exercise 5, Art. 192)

4. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots$. (exercise 2, Art. 192)

Divergent series for comparison.

1. $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \ (r \geqq 1)$. (exercise 1, Art. 192)

2. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$. (exercise 6, Art. 192)

EXERCISES

Show that each of the following series is convergent:

1. $1 + \frac{1}{4} + \frac{1}{2 \cdot 4^2} + \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} + \cdots$.

2. $1 + \frac{1}{3+1} + \frac{1}{3^2+1} + \frac{1}{3^3+1} + \cdots$.

3. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

4. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots$.

5. $\frac{1}{2^2+3} + \frac{1}{3^3+3} + \frac{1}{4^4+3} + \cdots$.

6. $1 + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \cdots$.

7. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

8. $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$.

9. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$.

10. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

11. $\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \cdots \ (p > 1)$.

12. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$.

Show that each of the following series is divergent:

13. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots$.

14. $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (0 < p,\ p < 1)$.

15. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$.

16. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$.

17. $\frac{1}{3} + \frac{2}{3} + \frac{2^2}{3} + \frac{2^3}{3} + \cdots$.

18. $\frac{1}{2 - \sqrt{2}} + \frac{1}{3 - \sqrt{3}} + \frac{1}{4 - \sqrt{4}} + \cdots$.

19. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$.

20. $\frac{1}{1 - \frac{1}{2}} + \frac{3}{2 - \frac{1}{3}} + \frac{3^2}{2^2 - \frac{1}{4}} + \cdots$.

Test each series by comparison or by Theorem I, Art. 189 to determine whether it converges or diverges.

21. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$.

22. $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} \cdots$.

23. $\frac{2^4}{2!} + \frac{3^4}{3!} + \frac{4^4}{4!} + \cdots$.

24. $\frac{\sqrt{2}}{2^2} + \frac{\sqrt{3}}{3^2} + \frac{\sqrt{4}}{4^2} + \cdots$.

25. $\frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 4}{4} + \cdots$.

26. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$.

27. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \cdots$.

28. $\frac{4}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \frac{6}{4 \cdot 5} + \cdots$.

194. Ratio test. Given an infinite series of positive terms

$$u_1 + u_2 + \cdots + u_n + u_{n+1} + \cdots. \tag{1}$$

Consider the ratio, $\frac{u_{n+1}}{u_n}$, of the $(n+1)$th to the nth term. Suppose that this ratio approaches a limit as $n \to \infty$. Call this limit R. In other words, suppose $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = R$. Then we may state the **ratio test** as follows:

(*a*) If $R < 1$, series (1) is convergent;
(*b*) if $R > 1$, series (1) is divergent;
(*c*) if $R = 1$, the test fails.

(*a*) $R < 1$. Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = R$, we can make $\frac{u_{n+1}}{u_n}$ differ from R by as small a number as we please. Hence, if we plot values of

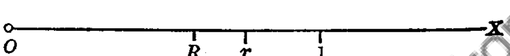

FIG. 47

$\frac{u_{n+1}}{u_n}$ on the line OX, Fig. 47, as n increases the points representing $\frac{u_{n+1}}{u_n}$ will concentrate about the point R. If n is taken large enough, they will lie to the left of the point r, where $R < r < 1$. For these values of n, we have

$$\frac{u_{n+1}}{u_n} < r, \qquad u_{n+1} < ru_n,$$

$$\frac{u_{n+2}}{u_{n+1}} < r, \quad u_{n+2} < ru_{n+1} < r^2u_n,$$

$$\frac{u_{n+3}}{u_{n+2}} < r, \qquad u_{n+3} < r^3u_n,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Since $r < 1$, the series

$$ru_n + r^2u_n + r^3u_n + \cdots = u_n(r + r^2 + r^3 + \cdots)$$

is convergent. But each term of the series

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots$$

is less than the corresponding term of the ru series. Hence, by Theorem I, Art. 191, the series $u_1 + u_2 + \cdots u_n + \cdots$ is convergent.

(b) $R > 1$. In this case the points representing $\dfrac{u_{n+1}}{u_n}$ will ultimately concentrate about the point R on the line OX in Fig. 48

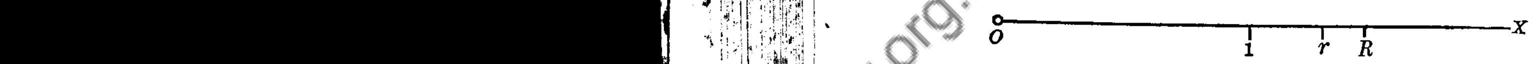

FIG. 48

and if n is large enough, they will lie to the right of the point r, where $1 < r < R$. Then

$$\frac{u_{n+1}}{u_n} > r,$$

or
$$u_{n+1} > ru_n;$$

and
$$u_{n+2} > r^2u_n,$$
$$u_{n+3} > r^3u_n,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Therefore, since the series

$$ru_n + r^2u_n + r^3u_n + \cdots$$

is divergent for $r > 1$,

the series $\quad u_1 + u_2 + u_3 + \cdots + u_n + \cdots$

is divergent (Theorem II, Art. 191).

(c) $R = 1$. If $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1$, this test fails. This is illustrated in the two series,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots,$$

and
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

The first has been shown to be convergent (exercise 4, Art. 192), the second divergent (exercise 6, Art. 192), but for each $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1$.

Example 1. Consider the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots.$$

Here,
$$u_{n+1} = \frac{n+1}{2^{n+1}}, \quad u_n = \frac{n}{2^n},$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{n}{2n} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}.$$

Hence, the series is convergent.

Example 2. Consider the series

$$\frac{2}{2^2} + \frac{2^2}{3^2} + \frac{2^3}{4^2} + \frac{2^4}{5^2} + \cdots.$$

Here,
$$u_{n+1} = \frac{2^{n+1}}{(n+2)^2}, \quad u_n = \frac{2^n}{(n+1)^2},$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{2^n} = 2\left(\frac{n+1}{n+2}\right)^2.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 2.$$

Hence, the series is divergent.

EXERCISES

Apply the ratio test to each series to determine its convergence or divergence. In case the ratio test fails, apply other tests, or use previous knowledge to answer the question of convergence.

1. $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots.$
2. $\frac{3^2}{2^3} + \frac{4^2}{2^4} + \frac{5^2}{2^5} + \cdots.$
3. $1 + \frac{1}{2} + \frac{1}{3} + \cdots.$
4. $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$
5. $1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \cdots.$
6. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots.$
7. $1 + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots.$
8. $\frac{1}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots.$
9. $\frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots.$
10. $\frac{1}{2+1} + \frac{2}{4+1} + \frac{3}{6+1} + \cdots.$
11. $\frac{1000}{1!} + \frac{1000^2}{2!} + \frac{1000^3}{3!} + \cdots.$
12. $1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots.$
13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$
14. $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots.$
15. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \cdots.$
16. $\frac{2}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2^2} + \frac{4}{5} \cdot \frac{1}{2^3} + \cdots.$
17. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$
18. $1 + \frac{2^4}{2!} + \frac{3^4}{3!} + \frac{4^4}{4!} + \cdots.$
19. $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots.$
20. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots.$
21. $1 + \frac{2!}{2^p} + \frac{3!}{3^p} + \frac{4!}{4^p}$, where p may have any real value.

SERIES WITH BOTH POSITIVE AND NEGATIVE TERMS

195. The following theorem will throw light on the convergence of series whose terms are not all of the same sign.

THEOREM. *An infinite series of real terms which are not all of the same sign is convergent if the series formed by making all the terms positive is convergent.*

After all the minus signs have been changed to plus signs, let the series be
$$u_1 + u_2 + u_3 + \cdots.$$
By hypothesis, this series is convergent and therefore has a limiting value S. The sum of the first n terms of this series is then less than S. Hence, the sum, S_n, of the first n terms of the original series is numerically less than S. Let these n terms consist of p positive and q negative terms. If P_p be the sum of the positive terms and N_q the sum of the negative terms, then
$$S_n = P_p - N_q.$$
But P_p and N_q are always less than S. Hence, by Art. 190, P_p and N_q approach fixed numbers P and N respectively as n increases without limit. Then
$$\lim_{n\to\infty} S_n = P - N, \text{ a definite number,}$$
and the series is convergent.

196. Generalized ratio test. The ratio test can readily be extended to series whose terms are not all of the same sign. Since a series of positive terms is convergent if
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1,$$
it follows, from the theorem just proved in Art. 195, that any series is convergent if the numerical value of $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$ is less than 1. That is, if
$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1.$$

If $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$, the nth term cannot approach zero as a limit, hence, by the corollary of Theorem I, Art. 189, the series is divergent. We may then write the ratio test for *any* infinite series
$$u_1 + u_2 + u_3 + \cdots$$
as follows:

If $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1$, *the series converges.*

If $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| > 1$, *the series diverges.*

If $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = 1$, *the test fails.*

Example: Test for convergence and divergence the series

$$1 - 2x + 3x^2 - 4x^3 + \cdots.$$

Here, $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)x^n}{nx^{n-1}}\right| = \left|\left(\frac{n+1}{n}\right)x\right|,$$

and $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|.$$

Hence, if x lies between $+1$ and -1, the series is convergent. For $|x| > 1$, the series is divergent. The interval between $+1$ and -1 is called

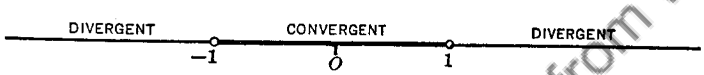

FIG. 49

the *interval of convergence* of the series, and is represented graphically by the heavy part of the line in Fig. 49. For the points 1 and -1 the test tells us nothing.

197. Alternating series. *A series whose terms are alternately positive and negative is convergent if each term is numerically less than the preceding term, and if the nth term approaches zero as a limit when n increases without limit.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n \pm \cdots,$$

where $$u_1,\ u_2,\ u_3,\ \cdots \text{ are positive},$$

and $$u_2 < u_1,\ u_3 < u_2,\ \cdots,$$

and where $$\lim_{n\to\infty} u_n = 0.$$

Let n be an even number. We may then write S_n in the form

$$S_n = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{n-1} - u_n).$$

Since each parenthesis contains a positive number, S_n is positive and increases as n increases. We may also write S_n in the form

$$S_n = u_1 - (u_2 - u_3) - \cdots - (u_{n-2} - u_{n-1}) - u_n.$$

Since the number within each parenthesis is positive,

$$S_n < u_1.$$

By the assumption of Art. 190, as n increases beyond bound, S_n approaches a limit S. But

$$S_{n+1} = S_n + u_{n+1},$$

hence, $\qquad \lim S_{n+1} = \lim S_n + \lim u_{n+1}.$

By hypothesis, $\quad \lim u_{n+1} = 0.$

Hence, $\qquad \lim S_{n+1} = \lim S_n = S,$

and the series is convergent.

198. Approximate sum of a series. In the case of some series, for example, a geometric progression, we are able to find exactly the limiting value of the sum of n terms as n increases without limit, but with many series we must be content to find an approximation to the limit, say correct to a certain number of decimal places.

Example: Calculate

$$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} + \cdots$$

correct to four decimal places.

$$\begin{array}{rlcrl}
1 & = 1.00000 & & & \\
\dfrac{1}{5!} & = 0.00833 & \qquad & -\dfrac{1}{3!} & = -\ 0.16667 \\
\dfrac{1}{9!} & = 0.00000 & & -\dfrac{1}{7!} & = -\ 0.00020 \\
 & 1.00833 & & -\dfrac{1}{11!} & = -\ 0.00000 \\
 & -\ 0.16687 & & & \\
 & 0.84146 & & & \ -\ 0.16687
\end{array}$$

To four decimal places then the sum is 0.8415. But the question arises as to just where we must stop adding terms. Even if $\frac{1}{11!}$ has no significant figure in the fifth place, we are dropping an infinite number of terms, a number of which when added together may affect the result materially. In the case of an alternating series, this question is easily answered by the

THEOREM. *The sum of the first n terms of a convergent, alternating decreasing series differs from the sum of the series by less than the* $(n+1)$*th term.*

Let S represent the limit of S_n, the sum of the first n terms as n becomes infinite, and R_n, the remainder. For n, even or odd, we have

$$| R_n | = u_{n+1} - u_{n+2} + u_{n+3} - \cdots.$$

From Art. 197, the sum of this alternating series, whose first term is u_{n+1}, is less than the first term. Hence,

$$| S - S_n | = | R_n | < u_{n+1}.$$

EXERCISES

Test the following series for convergence and divergence:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$

2. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots.$

3. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots.$

4. $\frac{2}{1 \cdot 3} - \frac{4}{3 \cdot 5} + \frac{6}{5 \cdot 7} - \cdots.$

5. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$

6. $\frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots.$

7. $\frac{\sqrt{3}}{2} - \frac{\sqrt{4}}{3} + \frac{\sqrt{5}}{4} - \frac{\sqrt{6}}{5} + \cdots.$

8. $\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots.$

9. $\frac{1}{2} - \frac{1}{2+a} + \frac{1}{2+2a} - \frac{1}{2+3a} + \cdots, \ (a > 0).$

10. $\frac{1}{1} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{4 \cdot 5 \cdot 6} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{5 \cdot 6 \cdot 7 \cdot 8} + \cdots.$

Compute the value of each series, correct to four decimal places:

11. $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots.$

12. $\frac{1}{10^2} - \frac{1}{20^2} + \frac{1}{30^2} - \frac{1}{40^2} + \cdots.$

13. $1 - \frac{2}{3!} + \frac{4}{5!} - \frac{6}{7!} + \cdots.$

14. $\frac{1}{10} - \frac{1}{20^2} + \frac{1}{30^3} - \frac{1}{40^4} + \cdots.$

199. Power series. The series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{1}$$

in which $a_0, a_1, a_2, \cdots$ are constants, is called a **power series.** Such a series clearly converges for $x = 0$. It may converge for all values of x, may diverge for all values of x except for $x = 0$, or it may converge for some values of x and diverge for others.

The generalized ratio test (Art. 196) is used to find the range of values of x for which such a power series as (1) converges.

When the values x for which a power series in x converges are represented graphically, they form an interval called the **interval of convergence.** The end points of the interval require special study to determine whether they are to be included in the interval of convergence.

If the series converges for all values of x, the interval is often said to extend from $-\infty$ to $+\infty$.

EXERCISES

Find the interval of convergence of the following series. Exhibit the results graphically.

1. $\frac{x}{2} + \frac{2^2x^2}{2^2} + \frac{3^2x^3}{2^3} + \cdots$.

Solution: $u_n = \frac{n^2x^n}{2^n}, \quad u_{n+1} = \frac{(n+1)^2x^{n+1}}{2^{n+1}}.$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2x}{2n^2} = \lim_{n\to\infty} \left(\frac{n^2}{2n^2} + \frac{2n}{2n^2} + \frac{1}{2n^2}\right)x = \frac{x}{2}.$$

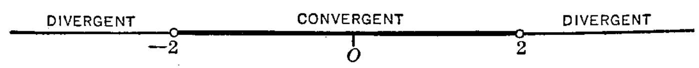

FIG. 50

The series is therefore convergent for $|x| < 2$ (Fig. 50).

2. $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$.

Solution: $u_n = \frac{x^n}{n}, \quad u_{n+1} = \frac{x^{n+1}}{n+1}.$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{nx}{n+1} = x.$$

Hence, the series converges if $|x| < 1$, that is if $-1 < x < 1$. With respect to the end points, 1 and -1, the series diverges if $x = 1$ (exercise 6, Art. 192). If $x = -1$, the series is an alternating decreasing series and thus converges. Hence, the interval of convergence is expressed by $-1 \leq x < 1$.

3. $1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots$.

4. $1 + \frac{2x}{3^2} + \frac{3x^2}{4^2} + \frac{4x^3}{5^2} + \cdots$.

5. $\frac{x^2}{2 \cdot 2^2} - \frac{x^4}{3 \cdot 2^4} + \frac{x^6}{4 \cdot 2^6} - \cdots$.

6. $1 + \frac{x}{2!} + \frac{2x^2}{3!} + \frac{3x^3}{4!} + \cdots$.

7. $1 + 2!x + 3!x^2 + 4!x^3 + \cdots (x \neq 0)$.

8. $\frac{1}{2} + \frac{3x}{2^2} + \frac{3^2x^2}{2^3} + \frac{3^3x^3}{2^4} + \cdots$.

By division expand the following fractions into series, and test for convergence.

9. $\frac{1}{1+x}$. **10.** $\frac{2}{1-2x}$. **11.** $\frac{1-x}{1+x}$. **12.** $\frac{(1-x)(1-2x)}{1+x}$.

The expansions of four functions of x are given below. Find the interval of convergence of each of the series.

13. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, if x is measured in radians.

14. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, if x is measured in radians.

15. $\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$.

16. $\text{Arc}\sin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots$.

17. From trigonometry we know that arc $\sin\frac{1}{2} = \frac{\pi}{6}$. Calculate the value of π to four decimal places from the series in exercise 16.

200. Binomial series. The power series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

is called the **binomial series.** If m is a positive integer, the series ends with the $(m+1)$th term and has been shown to be the expansion of $(1+x)^m$. If m is not a positive integer, the series is an infinite series, but it can be shown that it converges towards $(1+x)^m$ when x has any value which makes the series convergent. In other words, it can be shown that for these values of x, the binomial expansion holds for any exponent, integral or fractional, positive or negative.*

The binomial series is convergent for $|x| < 1$, and divergent for $|x| > 1$. For, we have, beginning with the term mx,

$$u_{n+1} = \frac{m(m-1)(m-2)\cdots(m-n)}{(n+1)!}x^{n+1},$$

$$u_n = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n,$$

$$\frac{u_{n+1}}{u_n} = \frac{m-n}{n+1}x,$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{m-n}{n+1}x = \lim_{n\to\infty}\frac{\frac{m}{n}-1}{1+\frac{1}{n}}x = -x.$$

Hence, from Art. 199, the series converges for $-1 < x < 1$.

* For a proof of the binomial expansion for any exponent, see Fine's *College Algebra*, p. 553.

In expanding $(b + x)^m$ for fractional or negative values of m, we may write it in the form $b^m\left(1 + \frac{x}{b}\right)^m$. The expansion will then proceed in powers of $\frac{x}{b}$, thus:

$$b^m\left(1 + \frac{x}{b}\right)^m = b^m\left(1 + m\frac{x}{b} + \frac{m(m-1)}{2!}\left(\frac{x}{b}\right)^2 + \cdots\right).$$

This series converges when $\left|\frac{x}{b}\right| < 1$, that is, the interval of convergence for the expansion of $(b + x)^m$ is the interval between $-b$ and $+b$ (Fig. 51).

FIG. 51

EXERCISES

Expand the following binomials to five terms and indicate the interval for which the expansion holds.

1. $(1 + x)^{-3}$.
2. $(1 + 2x)^{-\frac{1}{2}}$.
3. $(1 - 2y)^{\frac{1}{2}}$.
4. $(1 - 3y)^{-\frac{1}{2}}$.
5. $(2 - 3x)^{-3}$.
6. $\sqrt{1 + x}$.
7. $\sqrt[3]{1 + 3x}$.
8. $\dfrac{1}{\sqrt{2 - 7y}}$.
9. $\dfrac{1}{\sqrt{7 - 2x}}$.
10. $(3 - 4y^2)^{\frac{4}{3}}$.
11. $\dfrac{1}{\sqrt[3]{1 + x^3}}$.

Extract the following roots to four places of decimals by the binomial expansion.

12. $\sqrt[3]{65}$.

Solution:

$$\begin{aligned} \sqrt[3]{65} &= (4^3 + 1)^{\frac{1}{3}} = 4\left(1 + \frac{1}{4^3}\right)^{\frac{1}{3}} \\ &= 4\left(1 + \frac{1}{3}\cdot\frac{1}{4^3} - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{4^6} + \cdots\right) \\ &= 4(1 + 0.00521 - 0.00003 + \cdots) \\ &= 4.0207^+. \end{aligned}$$

13. $\sqrt{10}$.
14. $\sqrt{17}$.
15. $\sqrt{8}$.
16. $\sqrt[3]{9}$.
17. $\sqrt[3]{1.05}$.
18. $\sqrt[6]{63}$.
19. $\sqrt[3]{26}$.
20. $\sqrt[5]{0.99}$
21. $\sqrt[3]{0.341}$.

201. Logarithmic series. The power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

is called the **logarithmic series.**

Since $\qquad u_{n+1} = \pm \frac{1}{n+1} x^{n+1}, \; u_n = \mp \frac{1}{n} x^n,$

we have $\qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = -x,$

and the series is convergent for $|x| < 1$.

It will be shown in the calculus * that this series converges to $\log_e(1 + x)$ for any value of x for which the series is convergent, where e is the base of natural logarithms discussed in Art. 144. The series can then be used to find logarithms of numbers to the base e. Thus,

$$\log_e\left(\tfrac{3}{2}\right) = \log_e\left(1 + \tfrac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^4}{4} + \cdots.$$

The logarithmic series can be used to calculate logarithms of positive numbers less than 2. However, it converges so slowly that it is not well adapted to numerical computation.

FIG. 52

To derive a more convenient series for the calculation of natural logarithms, we proceed as follows: †

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Hence, $\qquad \log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

By subtraction,

$$\log_e(1 + x) - \log_e(1 - x) = \log_e\frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

* Townsend and Goodenough's *First Course in Calculus*, p. 326.

† For a more detailed discussion, see Osgood's *Introduction to Infinite Series*, pp. 23 and 44.

Let $$\frac{1+x}{1-x} = \frac{m+1}{m},$$

whence $$x = \frac{1}{2m+1}.$$

We have then

$$\log_e \frac{m+1}{m} = 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right),$$

$$\log_e(m+1) = \log_e m + 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right).$$

If $m = 1$, we have

$$\log_e 2 = 0 + 2\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5 \cdot 3^5} + \cdots\right) = 0.6931 + \cdots.$$

Letting $m = 2$,

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \cdots\right)$$
$$= 0.6931 + 0.4055 + \cdots = 1.0986 + \cdots.$$

In this way the logarithm of any number to the base e may be computed.

From Art. 144, if a is any positive number, we have

$$\log_{10} a = \frac{\log_e a}{\log_e 10} = \frac{1}{\log_e 10} \cdot \log_e a.$$

Hence, if we have computed the logarithm of a number to the base e, we can find its logarithm to the base 10 by multiplying by $\frac{1}{\log_e 10}$. To five significant figures, $\frac{1}{\log_e 10} = 0.43429$.

In computing a table of logarithms we need compute the logarithms of prime numbers only. The logarithms of composite numbers may then be found by means of the theorems on logarithms.

EXERCISES

1. Using the series for $\log_e(1 + x)$, compute $\log_e \frac{5}{4}$ to three places of decimals and then find $\log_{10} \frac{5}{4}$.

2. Calculate $\log_e 5$ to four significant figures.

3. Find $\log_e 9$ and $\log_e 10$ to four significant figures.

4. By computing the logarithms of 2, 3, 5, 7, construct a table of logarithms to the base 10 for the whole numbers 1 to 10, and verify by reference to a table.

ANSWERS TO ODD NUMBERED EXERCISES AND PROBLEMS

[The answers to even numbered exercises are omitted for reasons stated in the preface. Those of some odd numbered exercises are also intentionally omitted. The answers to even numbered exercises are available in a separate answer book.]

Art. 8. Page 12

1. $4y - x - 5.$
3. $-2t - 1.$
5. $17s - 3t.$
7. $6m + 1.$
13. $5.$
15. $3a - 6b + 2.$

Art. 10. Written Exercises. Page 16

1. $\dfrac{1}{(x + y)}.$
3. $\dfrac{1}{x + y + 1}.$
5. $\dfrac{-2}{a + b}.$
7. $\dfrac{2}{5}.$
9. $\dfrac{3y}{(x + y)^2}.$
11. $\dfrac{s + 2}{3(s + 3)}.$
13. $\dfrac{49}{7 - x}.$
15. $\dfrac{2b}{a + b}.$
17. $xy(x^2 + y^2).$
19. $\dfrac{1}{a}.$
21. $a(a + b).$

Art. 11. Page 18

1. $\dfrac{7}{10}.$
3. $\dfrac{a}{b}.$
5. $\dfrac{1}{x - 1}.$
7. $\dfrac{2y - 3x}{5y - 6x}.$
9. $\dfrac{4b - 3a}{b + a}.$
11. $\dfrac{3xy^2 + 2y}{x^2}.$
13. $\dfrac{50ab + 5}{a^2b}.$
15. $ac.$
17. $\dfrac{2x^2 - 3y}{x^2y}.$
19. $2a^2 + 3ab.$
21. $a - \dfrac{1}{4}.$
23. $\dfrac{7 + 2a}{18 + 5a}.$

Miscellaneous Exercises. Page 19

1. $17x - 3y.$
13. $\dfrac{m^2 + 4n^2}{mn}.$
15. $\dfrac{x(2x + 1)}{5 - 2x}.$
17. $\dfrac{1}{a - b}.$
19. $(5x + 4)\cdot 2 + y - 8 = 10x + y.$
21. \$975.61.
23. 23.64.
25. 0.716.
27. 30.
29. 60.

Art. 14. Written Exercises. Pages 23, 24

1. $-\dfrac{19}{16}$, 89, 109, $t^2 - 3t + 1.$
3. $A = \pi r^2.$
5. $\dfrac{4}{5}, 0, \dfrac{7}{8}, \dfrac{5}{8}, \dfrac{s^2 + 1}{s^2 + 2}.$
7. $3, -1, \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}, \dfrac{x + 2}{x}.$

Let
$$\frac{1+x}{1-x} = \frac{m+1}{m},$$
whence
$$x = \frac{1}{2m+1}.$$
We have then
$$\log_e \frac{m+1}{m} = 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right),$$
$$\log_e(m+1) = \log_e m + 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right).$$
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Art. 11. Page 18

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3. $\frac{a}{b}$.
5. $\frac{1}{x - 1}$.
7. $\frac{2y - 3x}{5y - 6x}$.
9. $\frac{4b - 3a}{b + a}$.
11. $\frac{3xy^2 + 2y}{x^2}$.
13. $\frac{50ab + 5}{a^2b}$.
15. ac.
17. $\frac{2x^2 - 3y}{x^2y}$.
19. $2a^2 + 3ab$.
21. $a - \frac{1}{4}$.
23. $\frac{7 + 2a}{18 + 5a}$.

Miscellaneous Exercises. Page 19

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13. $\frac{m^2 + 4n^2}{mn}$.
15. $\frac{x(2x + 1)}{5 - 2x}$.
17. $\frac{1}{a - b}$.
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29. 60.

Art. 14. Written Exercises. Pages 23, 24

1. $-\frac{19}{16}$, 89, 109, $t^2 - 3t + 1$.
3. $A = \pi r^2$.
5. $\frac{4}{5}, 0, \frac{7}{8}, \frac{5}{8}, \frac{s^2 + 1}{s^2 + 2}$.
7. $3, -1, \frac{\sqrt{2} + 1}{\sqrt{2} - 1}, \frac{x + 2}{x}$.

Art. 15. Page 26

5. $(-1, -5)$.

Art. 18. Page 31

3. -1. **5.** 0, 4. **7.** 4, 1. **9.** $0, +3, -3$.

Art. 20. Page 33

1. No.
3. Yes.
5. Yes.
7. Yes.
9. -3.
11. $\frac{15}{22}$.
13. $\frac{3}{2}$.
15. 3.
17. 5.
19. $-\frac{34}{11}$.

Art. 27. Page 41

1. $x = 3,\ y = 4$.
3. No solution.
5. An indefinite number of solutions.
7. $x = 2,\ y = -3$.
9. $x = -3,\ y = 2$.

Art. 28. Page 43

1. $x = 4,\ y = 3$.
3. $x = -2,\ y = 7$.
5. $x = 0.5,\ y = 1.1$.
7. $x = 0,\ y = 0$.
9. $x = 4,\ y = 3$.

Art. 29. Page 45

3. $x = -1,\ y = 3$.
5. $x = 0.1,\ y = -0.2$.
7. $x = \frac{3}{2},\ y = -\frac{5}{6}$.
9. $x = \frac{1}{8},\ y = -\frac{1}{5}$.

Art. 30. Page 46

1. $x = 1,\ y = 2,\ z = 3$.
3. $x = 7,\ y = \frac{2}{3},\ z = 1$.
5. $x = \frac{3}{2},\ y = -\frac{1}{2},\ z = \frac{5}{2}$.
7. $x = a,\ y = 2a,\ z = 3a$.
9. $x = 3,\ y = 2,\ z = -2$.

Art. 31. Page 47

3. 5. **5.** 336. **7.** $-avx - buy$.

Art. 32. Page 49

3. $x = 1,\ y = 2,\ z = 2$.
5. $x = 2,\ y = 3,\ z = -1$.
7. $x = -\frac{191}{3},\ y = 48,\ z = -\frac{58}{9}$.
9. $x = a + b,\ y = a - b,\ z = 2a$.
11. $x = 0,\ y = 2,\ z = 4$.

Miscellaneous Problems. Pages 50, 51

1. 48, 84.
3. 60 years, 24 years;
$\frac{mb(n-1) + na(m-1)}{m-n}$ years, $\frac{b(n-1) + a(m-1)}{m-n}$ years.

5. $3x^2 - 7x + 2$. **7.** 30°, 60°, 90°. **9.** 50 rods, 30 rods.

11. \$20,000 at $3\frac{1}{2}$ per cent, \$15,000 at 4 per cent.

13. \$60,000.

15. $\frac{P}{2a}, \frac{2acn - (a + c)P}{2a(c - b)}, \frac{(a + b)P - 2abn}{2a(c - b)}$.

17. Anal. geom. 83, algebra 93, trigonometry 88.

19. 92, 86, 83; 97, 87, 77.

21. 3 oz. 72 per cent silver, 5 oz. 84.8 per cent silver.

23. $\frac{p - cn}{t - c}$.

25. 5 min. $29\frac{1}{3}$ sec., 6 min. 45 sec.

27. $b = 761.4 - 0.0863h$.

29. $c = 0.036$, $b_0 = 999.5$.

33. $x = 2$.

35. $x = 2$, $y = 5$.

Art. 34. Page 55

1. 7. **3.** 10. **5.** 4, 3. **7.** $a^{12}b^5$. **9.** $\frac{1}{a^3}$. **11.** h^{n+3}.

13. $\frac{3a^3}{2bc}$. **15.** $-27m^6n^9$. **17.** $\frac{4m^2}{25x^2y^2}$. **19.** -243.

21. $a^{2n}x^{n-2}$. **23.** $a^{2n}b^{3n}$. **25.** $r^{n^2}s^{2n^2}$. **27.** a^4. **29.** $\frac{b^3}{2a^3}$.

31. $\frac{(x^2 - y^2)}{x^4 + y^4}$. **33.** $(a - 1)^n$. **35.** 1. **37.** 35.492.

Art. 39. Page 57

1. 8. **3.** 4. **5.** 27. **7.** 1. **9.** 27. **11.** 2. **13.** x. **15.** 0.5. **17.** 1.

Miscellaneous Exercises. Pages 58–60

1. $\frac{1}{16}$. **3.** $\frac{1}{27}$. **5.** $-\frac{1}{2}$. **7.** x^7. **9.** 2. **11.** $\frac{3}{5}$. **13.** $\frac{10}{9}$. **15.** 20. **17.** $\frac{4y^2}{x^2}$.

19. xy^{-3}. **21.** $(1.08)^{-20}$. **23.** $3x^{\frac{1}{2}}y^{-\frac{1}{2}}$. **25.** $a^2b^4c^3$. **27.** $\frac{x^{0.06}}{y^{0.14}}$.

29. $\frac{y^2}{x}$. **31.** $\frac{P^{0.47}}{q^{\frac{1}{3}}}$. **33.** $xy^2 + x^{\frac{1}{3}}y^{\frac{2}{3}}$. **35.** $a + 1$.

37. $m^{-2} - m^{-1} + 1$. **41.** $2 \cdot 3^{\frac{1}{4}}$. **43.** $5 \cdot 5^{\frac{1}{2}}$. **45.** $20 \cdot 2^{\frac{1}{2}}$. **49.** $(40)^{\frac{1}{3}}$. **51.** $(351)^{\frac{1}{3}}$.

53. $(192)^{\frac{1}{3}}$. **55.** x^2. **57.** $x^{\frac{7}{8}}$. **59.** $a^{9n^2}b$. **61.** 1250. **63.** 170.4.

65. $14 \cdot 10^{21}$. **67.** $3\frac{3}{4}, -\frac{1}{2}, 0, 3\frac{3}{4}$. **69.** $666 \cdot 10^{-10}$.

71. 0.000000005305.

73. 0.6867, 0.5896, 0.4861, 0.3934 microns.

Art. 41. Pages 62, 63

1. $\sqrt[3]{a^2}$. **3.** $2\sqrt{2x}$. **5.** $\sqrt[7]{(x + y)^6}$.

7. $\sqrt{pq}$. **9.** $\frac{1}{2}\sqrt[3]{2^2}$.

11. $a^{\frac{3}{4}}$. **13.** $a^{\frac{3}{2}}b^{\frac{5}{2}}c^2$. **15.** ab^2c^3.

17. $(a+b)^{\frac{1}{4}}$.

19. $\dfrac{a^{\frac{1}{3}}b^{\frac{2}{3}}c}{x^{\frac{5}{3}}y^{\frac{1}{3}}z^{\frac{4}{3}}}$.

21. $\dfrac{1}{a^2b^{\frac{2}{3}}}$.

23. $a^{\frac{5}{6}}$.

25. $\sqrt{75}$.

27. $\sqrt{252}$.

29. $\sqrt[5]{1000}$.

31. $\sqrt[3]{-375m^5n^3}$.

33. $\sqrt[3]{\dfrac{27am}{5}}$.

35. $\sqrt{(1+x)^3}$.

39. $\sqrt{10}$.

45. $\sqrt[12]{b^4}$, $\sqrt[12]{c^3}$.

47. $\sqrt[15]{3125}$, $\sqrt[15]{343}$.

49. $2\sqrt[6]{m^3}$, $\sqrt[6]{x^2y^5}$.

51. $\sqrt[12]{x^8y^8z^4}$, $2\sqrt[12]{x^3y^9z^6}$.

53. $2\sqrt{2}$.

55. $2ab\sqrt{3b}$.

57. $3\sqrt[3]{2}$.

59. $5\sqrt{5}$.

61. $\dfrac{2x^2\sqrt{6}}{9}$.

63. $\dfrac{7\sqrt{2}}{4}$.

65. $2\sqrt{3}$.

67. $\sqrt{a-b}$.

69. $-\dfrac{2}{3}$.

71. $2a^3\sqrt{7b}$.

73. $7a^nb^4c^2$.

Art. 43. Pages 64, 65

1. $12\sqrt{2}$.

3. $24\sqrt{5}$.

5. $8\sqrt[3]{3}$.

7. $\dfrac{43\sqrt{6}}{6}$.

9. $(3b+4ac^2+2b^2c)\sqrt{b}$.

11. $(ac+8b^2-5a^2b)\sqrt[4]{bc^2}$.

13. 0.

15. $\left(\dfrac{1}{y}-\dfrac{1}{x}\right)\sqrt{xy}$.

19. $(b+2ab^2-3b^3)\sqrt{2a}$.

21. $(1+2a-3b)\sqrt{3}$.

23. $18(\sqrt{3}-\sqrt{2})$.

Art. 44. Page 66

1. $15\sqrt{amn}$.

3. $40\sqrt{70}$.

5. $ab\sqrt[6]{a^2b}$.

7. $10a\sqrt[12]{a^{11}b^4c^3}$.

9. $6\sqrt[5]{5}$.

13. $ab\sqrt[4]{b}$.

15. $3\sqrt[4]{50}$.

17. $x^2yz\sqrt[20]{x^8y^2z}$.

19. $40a^2bc^2\sqrt[12]{a^3b^2}$.

21. $-3r^2sp\sqrt[6]{r^3s^4}$.

23. -2.

25. $6+2\sqrt{15}-24\sqrt{2}$.

27. $25-11\sqrt{5}$.

29. $59+4\sqrt{6}-6\sqrt{10}-12\sqrt{15}$.

31. $9a^{\frac{2}{3}}+12a^{\frac{1}{3}}b^{\frac{1}{3}}+4b^{\frac{2}{3}}$.

33. 2023.

35. $113+8\sqrt{30}-4\sqrt{15}-48\sqrt{2}$.

37. 0.

Art. 45. Pages 67, 68

1. $\sqrt{2}$.

3. $\sqrt{7}$.

5. 3.

7. $\sqrt[3]{ab}$.

9. 3.

11. $\dfrac{\sqrt[3]{abc^2d^2}}{cd}$.

13. $\dfrac{\sqrt[3]{xy^2}}{y}$.

15. $\sqrt{6}-4\sqrt{3}$.

17. $\dfrac{3\sqrt[3]{49}}{7}$.

19. $\dfrac{1}{b}$.

21. $\dfrac{\sqrt{a+b}\,(\sqrt{a}+\sqrt{b})}{a+b}$.

23. $-\dfrac{\sqrt{3}+2\sqrt{5}}{34}$.

25. $-\dfrac{7\sqrt{10}+63}{71}$.

27. $2+\sqrt{2}$.

29. $\sqrt{15}$.

31. 13.89.

33. 0.9537.

35. 0.2679.

37. 1.260.

Art. 46. Pages 69, 70

1. 1.
3. No solution.
5. -2.
7. -1 and 7.
9. $\frac{21}{4}$.
11. 6.
13. 0.
15. $s = \pm\sqrt{\frac{4A\sqrt{3}}{3}}$.
17. $i = \sqrt[t]{\frac{A}{P}} - 1$.
19. 1336.
21. 5.9 sec.

Art. 47. Page 71

1. $8i$.
3. $-2ia\sqrt{5}$.
5. $6bi$.
9. 1.
11. $x^2 + a^2$.
13. 7.
15. i.
17. 14.
19. -1054.
21. 0.

Miscellaneous Exercises and Problems. Pages 71–73

1. $\sqrt[6]{243}$.
3. $\frac{1}{2}\sqrt[6]{2}$.
5. $\frac{1}{3}\sqrt[6]{243}$.
7. $2(6^{\frac{1}{2}} + 3)$.
9. $5 + 2 \cdot 6^{\frac{1}{2}}$.
11. $5 + 3(12)^{\frac{1}{3}} + 3(18)^{\frac{1}{3}}$.
13. 0.
15. $(a + b)^{\frac{7}{12}}$.
17. $\frac{3}{4}\sqrt{3}$.
19. $\sqrt[6]{12}$.
21. $\frac{x}{y}$.
23. $(ab)^{\frac{1}{2}}$.
25. $5^{\frac{1}{3}}$.
27. 1.
29. $\frac{1}{2}\sqrt{6} + 1$.
31. $\frac{1}{2} \cdot 2^{\frac{1}{2}}$.
33. $\frac{26 - 7 \cdot 3^{\frac{1}{2}}}{23}$.
35. $a + 1 + \frac{1}{a}$.
37. 11.8.
39. 249.
41. 0.039.
43. 9.74.
45. 0.0059.
47. 322.
49. 45.17.

Art. 49. Page 75

3. $-2, -4$.
5. $\frac{1}{2}, -3$.
7. $n, -m$.
9. n if we consider the equation as a quadratic in s; s, if a quadratic in n.
11. $0, \frac{7}{3}$.

Art. 50. Page 77

1. $7, -1$.
3. $1, -\frac{2}{5}$.
5. $-10, -12$.
7. $\frac{3}{2}, -\frac{2}{3}$.
9. $\frac{1}{3}, -\frac{2}{3}$.
11. $2.5, -7$.
13. $4 + \sqrt{6}, 4 - \sqrt{6}$.
15. $2, -\frac{4}{3}$.
17. $3 + 4\sqrt{2}, 3 - 4\sqrt{2}$.
19. 1.
21. $x = \frac{m}{1 - e}$ or $\frac{m}{1 + e}$; $m = (1 + e)x$ or $(1 - e)x$; $e = \frac{m - x}{x}$ or $\frac{x - m}{x}$.
23. $b, \frac{2a^2 - b^2}{b}$.
25. $2a, -b$.
27. $a, b + 2$.
29. $\frac{n}{2}, -\frac{m}{3}$.
31. $1.9, -3.7$.
33. $3.7, 5.3$.

Art. 51. Pages 78, 79

3. $1, -1, 3, -3.$
5. $2, -2, 3, -3.$
7. $\frac{1}{4}, -\frac{1}{3}.$
9. $28.$
11. $\frac{5}{4}.$
13. $\frac{-1 \pm \sqrt{13}}{2}, \frac{-1 \pm i\sqrt{3}}{2}.$
15. $\frac{1}{4}, \frac{1}{5}.$
17. $2, -\frac{2}{3}.$
19. $2 \pm 2i, 2 \pm \sqrt{5}.$
21. $1, -2, -\frac{1 \pm i\sqrt{7}}{2}.$
23. $\frac{1}{2}, 2, \frac{1 \pm 3i\sqrt{11}}{10}.$
25. $-9.$

Art. 52. Page 80

3. $x^2 - 2x - 3 = 0.$
5. $x^2 + 4x + 3 = 0.$
7. $6x^2 + 6x - 1 = 0.$
9. $x^2 - 4x + 1 = 0.$
11. $x^2 - 2x + 5 = 0.$
13. $x^2 - \left(\frac{a^2 + b^2}{ab}\right)x + 1 = 0.$

Art. 54. Page 81

3. $k = 3$, or -1.
5. $k = 1, m = 0.$
7. $k = 1$ or -2.

Art. 56. Pages 82, 83

1. Real and unequal.
3. Real and equal.
5. Imaginary.
9. $k = \pm\sqrt{3}.$
11. $k = \frac{1}{3}.$
13. $k = 4.$
15. $k = 4$, or 0.
17. $k = \pm\frac{2}{3}\sqrt{3}.$
19. $\frac{8}{5}, 2.$
21. $-\frac{3}{2}, -\frac{7}{2}.$
23. $-\frac{2v_0}{g}, -\frac{2d}{g}.$
25. $k = -7.$
27. $k = -3.$

Art. 57. Page 84

11. The graph is concave downwards. The graph is symmetrical with respect to the y-axis. The graph passes through the origin.

Problems. Pages 84–87

1. 21, 22.
5. 3.73 sec.
7. 55.9 feet.
9. 4.23^+ sec.
11. 144.9 feet.
13. 45 miles per hour.
15. 1.303^- inches.
17. $\frac{a + b - \sqrt{a^2 - ab + b^2}}{6}; \frac{a}{6}.$
19. 0.56.

21. The graph; a parabola convex downwards, widens out, one root approaching 3, the other increasing without limit.

23. 16 inches.

25. 8 feet per second.

27. 27.1 square feet.

29. 5 or 0.

31. $n - n' = \dfrac{a \pm \sqrt{a^2 - 4T^2b}}{2Tb}$.

33. 1 part corn to 4 parts rye.

35. $\dfrac{1 \pm \sqrt{17}}{2}$ or 2.56^+, -1.56^-.

Art. 60. Pages 92, 93

1. (3, 2), (2, 3).

3. (7, 1), (− 7, − 1).

5. $(1, 1), \left(-\frac{19}{13}, -\frac{35}{13}\right)$.

7. $(3, 2), \left(\frac{3}{5}, -\frac{14}{5}\right)$.

9. (2, 0), (− 4, − 3).

11. $\left(\frac{3}{10}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{3}\right)$.

13. (4, 0), (0, 3).

15. (1.4, − 0.17), (0.036, − 6.7).

17. 640 yards, 484 yards.

19. 15, 16; − 16, − 15.

21. 82, 730.

23. $r > \sqrt{2}$, $r = \sqrt{2}$, $r < \sqrt{2}$.

25. Two points for any real value of a.

27. $b = \pm r\sqrt{1 + m^2}$.

29. (3, 4).

Art. 62. Page 95

3. (3, 2), (− 3, 2), (− 3, − 2), (3, − 2).

5. (4, 3), (4, − 3), (− 4, 3), (− 4, − 3).

7. $\left(\frac{9}{5}\sqrt{5}, \frac{2}{5}\sqrt{55}\right), \left(\frac{9}{5}\sqrt{5}, -\frac{2}{5}\sqrt{55}\right), \left(-\frac{9}{5}\sqrt{5}, \frac{2}{5}\sqrt{55}\right), \left(-\frac{9}{5}\sqrt{5}, -\frac{2}{5}\sqrt{55}\right)$.

9. (1.1, 2.3), (1.1, − 2.3), (− 1.1, 2.3), (− 1.1, − 2.3).

Art. 63. Page 97

1. (5, 3), (− 5, − 3).

3. (4, 1), (− 4, − 1), (14, − 4), (− 14, 4).

5. $(2, -3), (-2, 3), \left(\frac{29}{51}\sqrt{51}, \frac{2}{51}\sqrt{51}\right), \left(-\frac{29}{51}\sqrt{51}, -\frac{2}{51}\sqrt{51}\right)$.

7. $(6, 2), (-6, -2), (8i, 6i), (-8i, -6i)$.

9. $\left(\frac{6}{5}\sqrt{5}, \frac{6}{5}\sqrt{5}\right), \left(-\frac{6}{5}\sqrt{5}, -\frac{6}{5}\sqrt{5}\right), \left(\frac{6i}{7}\sqrt{7}, -\frac{12i}{7}\sqrt{7}\right), \left(-\frac{6i}{7}\sqrt{7}, \frac{12i}{7}\sqrt{7}\right)$.

11. $(1, 2), (-1, -2), \left(\frac{5}{7}\sqrt{7}, \frac{1}{7}\sqrt{7}\right), \left(-\frac{5}{7}\sqrt{7}, -\frac{1}{7}\sqrt{7}\right)$.

13. (10, 1.4), (− 10, − 1.4), (11, 17), (− 11, − 17).

15. 3 and 12.

Art. 64. Page 98

1. (3, 2), (2, 3), (− 3, − 2), (− 2, − 3).

3. $(2, -1), (-1, 2), \left(-1+\frac{1}{2}\sqrt{6}, -1-\frac{1}{2}\sqrt{6}\right), \left(-1-\frac{1}{2}\sqrt{6}, -1+\frac{1}{2}\sqrt{6}\right)$.

5. $s = 0, t = -4$; $s = -4, t = 0$; $s = 2.158, t = -1.158$; $s = -1.258$, $t = 2.158$.

Art. 65. Miscellaneous Exercises and Problems. Pages 99–102

1. $(5, -2), (-2, 5)$.

3. $(4, 4)$.

5. $\left(\frac{3}{2}\sqrt{2}, \sqrt{2}\right), \left(\frac{3}{2}\sqrt{2}, -\sqrt{2}\right), \left(-\frac{3}{2}\sqrt{2}, \sqrt{2}\right), \left(-\frac{3}{2}\sqrt{2}, -\sqrt{2}\right)$.

7. $(3, 6), (-3, -6), (6, 3), (-6, -3)$.

9. $\left(\frac{1}{13}, \frac{1}{8}\right), \left(-\frac{1}{8}, -\frac{1}{13}\right)$.

11. $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$.

13. $\left(\frac{1}{5}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{5}\right), \left(-\frac{1}{5}, -\frac{1}{3}\right), \left(-\frac{1}{3}, -\frac{1}{5}\right)$.

15. $\left(-\frac{3}{4}, -\frac{1}{2}\right), \left(\frac{5}{3}, -\frac{1}{3}\right), (6, 4), \left(-\frac{5}{11}, \frac{1}{11}\right)$.

17. $(2\sqrt{3}, \sqrt{3}), (-2\sqrt{3}, -\sqrt{3})$.

19. $(2, 3), (3, 2), (-4 + \sqrt{10}, -4 - \sqrt{10}), (-4 - \sqrt{10}, -4 + \sqrt{10})$.

21. $\left(\frac{1}{4}, \frac{1}{4}\right)$.

23. $(1, 36)$.

25. $(2, 1)$.

27. $\left(\frac{2}{5}, 2\right)$.

29. $\left(\frac{1}{2}, \frac{1}{3}\right), \left(-\frac{1}{2}, -\frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(-\frac{1}{3}, -\frac{1}{2}\right)$.

31. $(2, 4), (4, 2), (3 + \sqrt{21}, 3 - \sqrt{21}), (3 - \sqrt{21}, 3 + \sqrt{21})$.

33. $(9, 4)$.

35. $\left(\frac{1}{4}, \frac{1}{6}\sqrt{6}\right), \left(\frac{1}{4}, -\frac{1}{6}\sqrt{6}\right), \left(-\frac{1}{4}, \frac{1}{6}\sqrt{6}\right), \left(-\frac{1}{4}, -\frac{1}{6}\sqrt{6}\right)$.

37. $(1, 3, 5), (-1, -3, -5)$.

39. 70 and 130.

41. 85.

43. 120 rods and 160 rods.

47. 280 rods.

49. 25, 26.

51. 7, 9.

53. 7 feet and 24 feet.

55. 7 and 9.

57. $x^2 - 10x + 9 = 0$.

59. 40 rods, 40 rods.

61. \$1100, \$900; 7%.

63. 12, 6, and 4.

Art. 69. Page 107

1. $x > 6$.

3. $x < -1$.

5. $x > -1$.

7. $x < -1, x > 2$.

9. $\frac{1}{2} < x < \frac{5}{3}$.

11. $0 < x < \frac{1}{2}$; $x < -1$.

13. All real values of x.

15. $k < -4$ or > 4; $-4 < k < 4$

17. $x > 3$.

19. $3 - \sqrt{2} < x < 3 + \sqrt{2}$.

Art. 72. Pages 108, 109

1. $\frac{7}{2}$.

3. 8.

5. 20.

7. $-12, -9, \frac{15}{2}$.

9. $\frac{9}{2}, \frac{9}{2}, \frac{121}{7}, -\frac{121}{7}$.

15. $6\frac{2}{3}, 8\frac{1}{3}$.

17. $2\frac{2}{5}, 1\frac{4}{5}, 1\frac{1}{5}, \frac{3}{5}$.

19. $15\frac{3}{7}$ feet.

Art. 77. Pages 112–114

1. $V = ke^3$.
3. $V = kr^3$.
5. $A = \frac{k}{d^2}$.
7. 37.5 pounds.
9. $k = 1,\ z = \frac{xy}{w}$.
11. \$6666.67, \$13,333.33, \$26,666.67, \$53,333.33.
13. 800 lbs.
15. $L = \frac{kt^4}{l^2}$.
17. 400 tons.
19. The strength of a rectangular beam varies directly, as the product of the breadth, b, and the depth, d, and inversely as the length, l.
21. 177.1 ft.
23. 225.4 ft. per sec., 788.9 ft.
25. 0.55 sec.
27. 5.33^+ sec.
29. 48 cu. in.
31. 47.43 miles per hour.

Art. 81. Pages 116, 117

3. 26, 33, 40.
7. $l = 46,\ s = 288$.
9. $l = -\frac{15}{2},\ s = -18$.
11. $d = \frac{78}{11},\ s = 396$.
13. $a = -20,\ d = 4$.
15. 2500.
19. $6\frac{1}{3},\ 8\frac{2}{3},\ 11,\ 13\frac{1}{3},\ 15\frac{2}{3}$.
21. $\frac{2x + y}{3},\ \frac{x + 2y}{3}$.

Art. 85. Pages 118, 119

1. $\frac{2}{3},\ 1,\ \frac{3}{2},\ \frac{9}{4},\ \frac{27}{8},\ \cdots$
5. $l = -39{,}366,\ s = -29{,}524$.
9. 381.
11. $a = \frac{1}{16{,}807},\ s = \frac{137{,}257}{16{,}807}$.
13. $33\frac{1}{3}$, 10.
15. $32x - \frac{y}{81},\ 119x - \frac{547y}{81}$.
17. 125.
19. 121.

Art. 88. Page 121

1. $\frac{3}{2}$.
3. 18.
5. 34.3.
7. $\frac{10}{11}$.
9. $\frac{70}{99}$.
11. $\frac{367}{495}$.

Art. 90. Page 122

3. $\frac{6}{5},\ \frac{3}{2};\ \infty,\ -6$.
5. $\frac{4}{3}$, 1.
7. $\frac{2ab}{a + b}$.

Problems. Pages 122–124

1. 747.
3. 84 feet.
5. Between 185 and 190 feet per second.
7. 122.853.
9. 2900 feet.
11. \$852.
13. $n(n + 1)$.
15. $\frac{1}{x}$.
17. 1.
21. $\frac{109}{27}$.
23. 1.38 qt.
25. 3, 8.

Art. 92. Page 128

1. 504. **3.** 30. **5.** 84. **7.** $r^2 - r$.

Art. 94. Pages 130, 131

9. $a^6 + 6a^5\sqrt{b} + 15a^4b + 20a^3b\sqrt{b} + 15a^2b^2 + 6ab^2\sqrt{b} + b^3$.

11. $\frac{12}{e} + \frac{40}{e^3} + \frac{12}{e^5}$.

13. $x^4 - 8x^{\frac{7}{2}}y^{\frac{1}{2}} + 28x^3y - 56x^{\frac{5}{2}}y^{\frac{3}{2}} + 70x^2y^2 - 56x^{\frac{3}{2}}y^{\frac{5}{2}} + 28xy^3 - 8x^{\frac{1}{2}}y^{\frac{7}{2}} + y^4$.

15. $a^9 + 6a^{\frac{49}{6}} + 15a^{\frac{22}{3}} + 20a^{\frac{13}{2}} + 15a^{\frac{17}{3}} + 6a^{\frac{29}{6}} + a^4$. **17.** 32.

19. $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

21. $\frac{x^8}{81} + \frac{4x^6}{27} - \frac{8x^3}{27} + \frac{2x^4}{3} + \frac{4x^2}{3} - \frac{8x}{3} + 1 - \frac{8}{x} + \frac{8}{3x^2} - \frac{8}{x^3} + \frac{16}{x^4} + \frac{24}{x^6} - \frac{32}{3x^7}$
$- \frac{32}{x^9} + \frac{16}{x^{12}}$.

23. $a^{\frac{13}{2}} - \frac{26}{3}a^{\frac{9}{2}} + \frac{104}{3}a^{\frac{5}{2}}$.

25. $105x^{\frac{11}{2}} + 15x^{\frac{13}{2}} + x^{\frac{15}{2}}$.

27. $-14{,}080a^9b^3$.

29. $5670x^8y^8$.

31. $\frac{-13!}{7!\,6!}x^6y^6a^{\frac{21}{2}}$.

33. $-35a^{\frac{6}{5}}, +35a^{\frac{8}{5}}$.

35. 941,480,149,401.

37. 0.885842380864.

39. 4.177.

Art. 97. Page 135

3. $r = 5,\ \theta = \arctan\left(-\frac{3}{4}\right)$.

5. $r = \sqrt{10},\ \theta = \arctan(-3)$.

7. $r = 1,\ \theta = -90°$.

9. $r = 7,\ \theta = -90°$.

11. $r = 6,\ \theta = 0°$.

13. $r = \frac{1}{4}\sqrt{13},\ \theta = \arctan\left(-\frac{2}{3}\right)$.

17. $-1 + i\sqrt{3}$.

19. $-\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$.

21. $-\frac{5}{2}i$.

23. $x = 4,\ y = 2$.

25. $x = 0,\ y = -1$ or
$x = \frac{2}{3},\ y = -\frac{1}{3}$.

Art. 98. Page 136

1. $8 + 3i$.

3. $4 - i$.

5. $\frac{1}{2} + \frac{1}{2}i$.

7. 8.

9. $-1 + 4i$.

Art. 100. Page 138

3. $6 + 6i\sqrt{3},\ \theta = 60°,\ r = 12$.

5. $-8,\ \theta = 180°,\ r = 8$.

7. $-2 - 2i,\ \theta = 225°,\ r = \sqrt{8}$.

9. $-10,\ \theta = 180°,\ r = 10$.

11. $3 - 3\sqrt{3}i,\ \theta = 300°,\ r = 6$.

Art. 102. Pages 139, 140

3. $16(-1 + i)$.
5. 1.
7. 16.
11. $\pm(\sqrt{3} + i)$.
13. $3(\cos x + i \sin x)$ where $x = 20°, 140°$, or $260°$.
15. $\cos x + i \sin x$ where $x = 7°, 127°$ or $247°$.
17. $-i, \pm\frac{\sqrt{3}}{2} + \frac{i}{2}$.
19. $\pm 1, \pm i$.
23. $2, -1, \pm i\sqrt{3}$.
25. $1, \cos x + i \sin x$ where $x = 72°, 144°, 216°$ or $288°$.
27. $\pm 1, \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Art. 103. Page 141

3. $\frac{i}{2}$.
5. $2 - 2i$.
7. $\frac{1}{4} - \frac{1}{4}i$.
9. $\frac{1 - i}{2}$.
11. $\frac{1 + i}{2}$.
13. $\frac{2 - 3i}{13}$.
15. $\frac{1 + 2i}{5}$.

Art. 107. Page 144

1. 2.
3. 18.
5. -3.
9. $x - 2, x^3 + 2x^2 + 4x + 8$.

Art. 109. Pages 146, 147

3. Quotient: $x^2 + 3x - 6$. Remainder: 0.
5. Quotient: $3x^2 + 2x + 4$. Remainder: 3.
7. Quotient: $x^3 - 4x^2 + 16x - 65$. Remainder: 257.
9. 12, 3, 43.

Art. 110. Page 148

3. Between -4 and -3.5, between 0 and 0.3, between 3 and 3.5.
5. Zeros at $-2, -1, 2, 3$.
7. Zero between -1.5 and -1, between -1 and -0.5, between 0 and 0.5 and between 2.5 and 3.

Art. 113. Pages 150, 151

1. Single roots: $2, -1$; double root: 3.
3. Single roots: $1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$.
5. Single root: 3; root of multiplicity $5 : \frac{1}{4}$.
9. (a) $x^3 - 10x^2 + 31x - 30 = 0$.
   (b) $x^3 - 5x^2 + 5x + 3 = 0$.
   (c) $x^2 - 2x + 5 = 0$.

Art. 117. Pages 155, 156

1. $y^3 + 10y^2 - 175y - 125 = 0.$
3. $y^4 - 10y^2 + 3y - 2 = 0.$
5. $y^3 - 6y^2 + 3y - 270 = 0.$
7. $y^3 + 36 = 0.$
9. $y^3 - 3y^2 + 10y + 12 = 0.$
11. $y^3 - 3y^2 + 15y - 26 = 0.$

Art. 118. Pages 157, 158

3. One neg., two imag.
5. One pos., four imag.
7. One pos., two imag.
9. One neg., two imag.
19. Three pos.
11. One pos., and $n - 1$ imag.
13. One neg., two pos.
17. Two pos., one neg.
21. Two pos., two neg.

Art. 120. Page 159

1. $b = 6,\ b_1 = -1.$
3. $b = 7,\ b_1 = -6.$
5. $b = 6,\ b_1 = 0.$
7. $b = 2,\ b_1 = -1.$
11. $-1. \cdots.$
13. $1. \cdots.$
15. $0. \cdots,\ 1. \cdots,\ 2. \cdots.$
17. $2. \cdots.$

Art. 121. Page 162

5. $1,\ 1,\ \pm 2.$
7. $6,\ -3,\ -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}.$
9. $\frac{1}{2},\ \frac{7}{4} \pm \frac{3}{4}\sqrt{5}.$
11. $-\frac{2}{3},\ -1 \pm \sqrt{2}.$
13. $\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{3}.$
15. $-\frac{1}{2},\ \pm i.$
17. $1,\ \frac{1}{2},\ \frac{1}{2}.$
19. No rational root.
21. $1,\ 2,\ -2,\ 3,\ 4.$

Art. 122. Page 164

1. $2.82^-.$
3. 1.62.
5. 4.51.

Art. 123. Page 166

3. $y^3 + 9y^2 - 90 = 0.$
5. $2y^4 - 3y^2 + 4y - 5 = 0.$
7. $y^3 + 5y^2 + 6y = 0.$
9. $y^3 + 7.1y^2 + 8.47y - 0.207 = 0.$

Art. 126. Pages 171–173

1. 1.20.
3. $5.24^-.$
5. 2.48.
7. 0.13.
9. 2.90.
31. 11.07.
33. 0.259.
11. $-3.40.$
13. 3.98.
15. $-3.$
17. $1.88,\ -0.35,\ -1.53.$
19. $3.01,\ 0.63,\ -2.02,\ -0.95.$
21. $2.36^-,\ 2.69,\ -2.05.$
23. 6.17.
25. 0.606.
27. 0.32, 0.64.
29. 4.00 per cent.
35. $2\sqrt[3]{\frac{292}{3}} - 8 = 1.20,\ \frac{5}{2}\sqrt[3]{\frac{292}{3}} - 10 = 1.50,\ 3\sqrt[3]{\frac{292}{3}} - 12 = 1.80$

Art. 127. Pages 174, 175

1. $x^3 - 2x^2 - x + 2 = 0.$
5. $1,\ x^3 - 4x^2 + 5x - 2 = 0.$
9. $2y^4 - 23y^2 - 25y + 6 = 0.$
13. $-1,\ 2,\ 5.$
3. $x^3 - 9x^2 + 23x - 15 = 0.$
7. $-2,\ x^4 - 5x^2 + 4 = 0.$
11. $2,\ 2,\ -2.$
15. $1,\ 2,\ 4,\ c = 14.$

Art. 130. Page 180

3. $-1, \frac{3}{2} \pm \frac{i\sqrt{3}}{2}$.

5. $-3, w, w^2$.

7. $\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \frac{1}{2}(-3 \pm i\sqrt{3})$.

9. $3, -1 \pm i$.

Art. 132. Page 182

1. $3, 3, 0, 1, \frac{1}{2}$.

3. $7, -4, \frac{1}{9}, 10, 0$.

Art. 133. Pages 183, 184

3. $\frac{1}{2}\log 7 - 2\log 2 - \frac{1}{3}\log 5$.

5. $\frac{1}{6}\log 5 - \frac{1}{2}\log 2 - \frac{1}{2}\log 3$.

7. $\log 2 + \frac{1}{3}\log 3$.

9. $3\log 2 + \log 7$.

11. 1.3222.

13. 1.7993.

15. 0.6990.

17. -0.1505.

19. -0.2922.

21. 0.8005.

23. 0.7398.

25. 2.3122.

Art. 139. Pages 188, 189

1. 71.4 means some number between 71.35 and 71.45; 71.40 means some number between 71.395 and 71.405; 71.400 means some number between 71.3995 and 71.4005.

3. 3.1415926536; 3.141592654; 3.14159265; 3.1415926; 3.141593; 3.14159; 3.1416; 3.142; 3.14; 3.1.

5. 60.5.

7. 5.4248125 and 5.9259375; 5.7.

9. 192.43^- and 193.18^-; 193.

Art. 143. Page 194

1. 86780.

3. 2692.

5. 13.16.

7. 0.0002439.

9. 0.7072.

11. -0.9648.

15. 0.00008254.

17. 1.585.

19. 424.2.

21. 0.9550.

23. 2.565.

25. -7.092.

27. 38.15.

29. 4.502.

31. 1698 pounds.

33. 1.084 sec.

35. 13.33; 27.19.

37. 108.3.

39. 1,476,000.

41. 1.246.

43. 177.5.

45. (1) 100,100; more accurate value 100,081;
(2) 85,450; more accurate value 85,442.

Art. 144. Page 198

3. 1.431.

5. 1.585.

7. 0.6131.

9. 3.215.

11. -1.585.

13. -0.8997.

Art. 146. Pages 200, 201

5. 0.4306.
7. ± 1.488.
9. 21.54.
11. 1000 or 0.01.
13. 2.
15. $\frac{1}{9}$.
17. $\frac{\log[(r-1)s+a]-\log a}{\log r}$.
19. $x = 7.925$, $y = -5.829$.
21. 0; ± 1.32.
23. $k = 0.00003776$.
25. $k = 0.126$.

Art. 149. Pages 204, 205

1. $1344 to the nearest dollar.
3. $1347 to the nearest dollar.
5. $219, $39.
7. $1070 to the nearest dollar.
9. 23.2 years.
11. 25 digits.
15. (*a*) $2653, (*b*) $2685, (*c*) $2718.

Art. 150. Page 207

3. $3439, $2559.
5. $6463.
9. $1288.

Art. 154. Pages 210, 211

1. 210; 840.
3. 113,400.
5. 11,352.
7. 80,640.
9. 12,144.
11. 14,400.
13. 17.

Art. 158. Pages 212, 213

1. 56.
3. 280.
5. 255.
7. 18.
9. 5.
11. 4.
13. 1512.
15. 462.
19. 26.
21. 63.
25. $(n-2)(n^2-4n+6)$.
27. 1225.

Art. 162. Page 216

1. $\frac{1}{2}$.
3. 0.45.
5. $\frac{9}{20}$.

Art. 164. Pages 217, 218

1. $13.50.
3. $\frac{4}{9}, \frac{1}{3}, \frac{2}{9}, \frac{5}{9}$.
5. 0.5136.
9. $\frac{30}{91}$.
11. $50.
13. $\frac{1}{24}$.
15. 0.216.

Art. 165. Page 220

1. $\frac{1}{2}$.
3. $\frac{1}{12}$.
5. $\frac{1}{2}$.
7. $\frac{7}{9}$.
9. $\frac{1}{30}$.
11. $\frac{8}{27}$.

Art. 166. Page 222

1. $\frac{3}{8}, \frac{11}{16}$.
3. $\frac{105}{512}, \frac{53}{64}$.
5. (1) 0.108; (2) 0.994.
11. $\frac{1}{7}$.
13. $40.
15. (1) $\frac{1}{8}$, (2) $\frac{25}{216}$.
17. The latter.
19. $\frac{4651}{7776}$.

Art. 168. Page 226

1. $\frac{1}{x} - \frac{1}{x+1}$.

3. $\frac{2}{3x+1} - \frac{1}{2x-1}$.

5. $\frac{5}{1-5x} - \frac{1}{x+5}$.

7. $x - \frac{1}{x-2} + \frac{1}{x+2}$.

9. $\frac{2}{x+1} + \frac{3}{x-2} - \frac{4}{x+2}$.

11. $\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3}$.

Art. 169. Page 227

1. $\frac{-1}{4(x+3)} + \frac{1}{4(x+1)} + \frac{3}{2(x+1)^2}$.

3. $\frac{2}{x} + \frac{1}{x^2} + \frac{3}{x-1}$.

5. $\frac{1}{x} + \frac{1}{3x-4} + \frac{1}{(3x-4)^2}$.

7. $1 + 2x - \frac{3}{x} + \frac{4}{x^2} - \frac{5}{x+1}$.

9. $\frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{x-1} + \frac{1}{(x-1)^2}$.

Art. 170. Page 228

1. $\frac{2x-3}{x^2+x+1} + \frac{1}{x+1}$.

3. $\frac{1}{x} - \frac{x}{x^2+5}$.

5. $\frac{1}{x+1} - \frac{x-2}{x^2-x+1}$.

7. $\frac{1}{3(x^2+2)} + \frac{1}{3(2x^2+1)}$.

9. $x + \frac{x-1}{x^2+1} + \frac{2x+3}{x^2+x+1}$.

Art. 171. Page 229

1. $\frac{2}{x} + \frac{1}{(x^2+1)^2}$.

3. $\frac{1}{1-x} + \frac{2x+3}{x^2+x+1} + \frac{x}{(x^2+x+1)^2}$.

5. $\frac{x+2}{x^2+1} - \frac{x}{(x^2+1)^2}$.

7. $\frac{1+x}{1+x^2} + \frac{1-x}{(1+x^2)^2} + \frac{2x-1}{1-x+x^2}$.

9. $\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{x-1}{x^2+1} + \frac{x+1}{(x^2+1)^2}$.

Art. 174. Page 235

1. 168.

3. 0.

Art. 175. Page 237

3. -35.

5. 16.

7. 84.

9. -99.

11. $ab(a-b)(a-1)(1-b)$.

13. $a(a-b)(b-c)$.

Art. 179. Pages 243, 244

1. 2, 1.
3. $\frac{1}{2}, \infty, 1$.
5. 2, -1, 3, -4.
9. Inconsistent.
11. 2, 1.
15. Inconsistent.

Art. 187. Page 253

1. 6.
3. 27.
5. $\frac{-3a}{2}$.
7. 60.
9. -9.
11. 4.
13. $\frac{5}{3}$.

Art. 189. Page 256

1. $1 + \frac{1}{2} + \frac{1}{3}, \frac{1}{n+1}$.
3. $-x + \frac{x^2}{2!} - \frac{x^3}{3!}, \frac{(-1)^{n+1}x^{n+1}}{(n+1)!}$.
5. $\frac{1}{\sqrt{n}}$.
7. $\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$.

Art. 193. Page 262

21. Divergent.
23. Convergent.
25. Divergent.
27. Divergent.

Art. 194. Page 265

1. Convergent.
3. Divergent.
5. Divergent.
7. Convergent.
9. Convergent.
11. Convergent.
13. Divergent.
15. Divergent.
17. Convergent.
19. Convergent.
21. Divergent.

Art. 198. Page 269

1. Convergent.
3. Convergent.
5. Convergent.
7. Convergent.
9. Convergent.
11. 0.6321.
13. 0.6988.

Art. 199. Pages 270, 271

3. Convergent for $|x| < 2$.
5. Convergent if $-2 \leqq x \leqq 2$, diverges if $x < -2$ or if $x > 2$.
7. Divergent for all values of x for which the series is defined.
9. Convergent for $|x| < 1$.
11. Convergent for $|x| < 1$.
13. Convergent for all values of x.
15. Convergent for all values of x.
17. 3.1416.

Art. 200. Page 272

1. $1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots, |x| < 1$.
3. $1 - y - \frac{1}{2}y^2 - \frac{1}{2}y^3 - \frac{5}{8}y^4 - \cdots, |y| < \frac{1}{2}$.
5. $\frac{1}{8}\left(1 + \frac{9}{2}x + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \frac{1215}{16}x^4 + \cdots\right), |x| < \frac{2}{3}$.

7. $1 + x - x^2 + \frac{5}{3}x^3 - \frac{10}{3}x^4 + \cdots, |x| < \frac{1}{3}.$

9. $\frac{\sqrt{7}}{7}\left(1 + \frac{1}{7}x + \frac{3}{98}x^2 + \frac{5}{686}x^3 + \frac{5}{2744}x^4 + \cdots\right), |x| < \frac{7}{2}.$

11. $1 - \frac{1}{3}x^3 + \frac{2}{9}x^6 - \frac{14}{81}x^9 + \frac{35}{243}x^{12} - \cdots, |x| < 1.$

13. 3.1623. **15.** 2.8284. **17.** 1.0164. **19.** 2.9625^-. **21.** 0.6986.

Art. 201. Page 274

1. $\log_e \frac{5}{4} = 0.223, \log_{10} \frac{5}{4} = 0.097^-.$ **3.** 2.1972, 2.3026.

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